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STRATEGIC LASER COMMUNICATIONS UPLINK ANALYSIS

The Optical Sciences Company

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STRATEGIC LASER COMMUNICATIONS
UPLINK ANALYSIS

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The work presented in this report is concerned with two distinct matters. Each of these is covered in one of the two chapters of this volume. Chapter 1 treats the subject of anisoplanatism in ordinary adaptive optics. The subject of Chapter 2 is the development of an estimate of pulsed laser backscatter signal from the upper atmosphere.

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In certain remote sensing applications, it is of interest to determine the power backscattered from air molecules when a pulsed laser beam propagates through the atmosphere. Our primary concern in Chapter 2 is the evaluation of the interaction between the laser pulse and the atmosphere at altitudes above the region where turbulence effects are appreciable. The analysis indicated that a significant fraction of the laser pulse is backscattered into a 1 m^2 detector if the pulse is observed as it propagates from an altitude of 20 km to an altitude of 30 km. Analysis and numerical results are presented.

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Chapter I

Anisoplanatism in Adaptive Optics Systems

1.1 Introduction

The use of adaptive optics for correction of the effects of atmospheric turbulence, when controlled by the technique which has become known by the name of "phase reversal" or "phase conjugation", relies on the availability of a suitable reference beacon.[1] The value of the turbulence induced perturbations of the wavefront of this reference signal as measured at the aperture of the adaptive optics system is assumed to provide a valid indication of the turbulence effects associated with the propagation path of interest. If the adpative optics is part of a compensated imaging system then it follows directly, while if the adaptive optics is part of a laser transmitter system it follows from consideration of reciprosity[2], that if the adaptive optics can apply a correction which corresponds to distorting an undistorted wavefront so that it is appropriately inverse* to the reference signals wavefront distortion, then the adaptive optics system will perform in a diffraction limited manner — providing that the propagation path for the reference signal exactly matches the propagation path for which diffraction limited system operation is desired.

This last point is of some significance since there are cases of interest for the application of adaptive optics systems for which the angular separation, $\vec{\vartheta}$, between the direction of the propagation path over which the adaptive optics system is to provide diffraction limited performance, and the direction of the propagation path along which the reference signal arrives at the adaptive optics system's aperture, is not zero. In such cases we have to ask whether or not the angular separation, $\vec{\vartheta}$, is large enough to be of noticeable consequence in the sense of significantly degrading the performance of the adaptive optics system. If the magnitude of the angle, $\vec{\vartheta}$, is small enough that there is no significant degradation, then we say that the angle "lies within the isoplanatic

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* The reader is cautioned here that exactly what constitutes the "appropriate inverse" is not uniquely defined when the power density of the beacon signal varies across the aperture, due to turbulence effects. For the compensated imaging system and for the laser transmitter, the correction for such power density variations are distinctly different, as we shall see.

patch".* If, however, the angular separation, $\vec{\vartheta}$, has a large enough magnitude that there is a significant amount of degradation of the performance of the adaptive optics system, then we would say that there is an anisoplanatism problem.

In this chapter we shall concern ourselves with the task of developing a set of theoretical results that will quantify these considerations. We shall treat two classes of adaptive optics systems — an imaging system and a laser transmitter. For the former we shall be interested in developing a relationship between the mean value of the turbulence degraded optical transfer function of the imaging system and the angular separation, $\vec{\vartheta}$, between the directions from the center of the system's entrance aperture to the reference signal's source and to the object to be imaged. For the latter our interest will lie in the task of relating the turbulence degraded antenna gain of the laser transmitter to the angular separation, $\vec{\vartheta}$, between the direction from the center of the system's exit aperture to the reference signal's source and the direction to the laser transmitters' aim-point.

A major point of interest in the analysis we shall be presenting relates to the treatment of the variations across the aperture of the beacon signal's power density. In the customary consideration of an adaptive optics system such variations are ignored. The adaptive optics applies a correction which is the inverse of the real turbulence induced phase shift sensed on the reference signal's wavefront. When $\vec{\vartheta}$ is identically equal to zero, and if there is no power density variation such operation of the adaptive optics will result in

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* The concept of isoplanatism and of an isoplanatic patch was first introduced in the analysis of the resolving power of lenses, as a basis for stating the assumption needed to justify working with Fourier transforms and thus introducing OTF and MTF concepts. The assumption was, in essence, that the isoplanatic patch size was large enough that it could be considered to be infinite. The term "isoplanatism" as it was used in discussing a lens had reference to the size, shape, and orientation of the image of a point source, and thus was only indirectly (and not uniquely) related to the wavefront aberration. Nonetheless, the terms "isoplanatism", "anisoplanatism", and "isoplanatic patch" seem clearly appropriate to denote the concepts we have used them to name in this chapter.

diffraction limited performance. However, if there is some nontrivial power density variation the performance will be less than diffraction limited. The origin of this degradation is perhaps most easily indicated by giving the name "random apodization" to the power density variation. Apodization, i. e., the variation of the the transmission across the aperture of an optical system, will effect the resolution of an imaging system and the antenna gain of a laser transmitter. In general the effect of apodization is the reduction of side-lobe levels at the expense of resolution or antenna gain. When the apodization is randomly "chosen"(by the turbulence) we may expect the reduction in resolution or antenna gain to be rather substantial for the amount of apodization, i. e., for the amount of power density variation.

An ideal adaptive optics system should be able to compensate for not only the real part of the turbulence induced phase variations but also for the imaginary part, i. e., for the power density variations.* When it does this the systems performance should, it would seem, revert to the diffraction limited value, when there is no anisoplanatism problem, i. e., when $\vec{\vartheta}$ equals zero. In the following analysis we shall see that this is indeed exactly the case for the adaptive optics imaging system. For the adaptive optics laser transmitter we shall find that performance cannot, in general, be made exactly equal to the diffraction limited value — and that rather surprisingly, under some circumstances the performance can actually exceed the diffraction limited value of antenna gain. Moreover, we shall find that the nature of the random apodization correction is fundamentally different for the laser transmitter from what it is for the imaging system.

In the next section we shall develop expressions for the diffraction limited performance of an imaging system and of a laser transmitter. Along with that, this section will also consider adaptive optics performance when

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* We consider it to be outside the scope of this work to discuss techniques by means of which adaptive optics compensation for random apodization could be achieved.

$\vec{\vartheta}$ equals zero, i. e., when there is no anisoplanatism. In doing this we will necessarily also consider in this section the control rules required for the random apodization correction part of an adaptive optics system. In the section after that we shall treat the statistics of propagation through turublence, developing formulas for the covariance of the real and imaginary parts of the complex phase variations induced by the turbulence. These results will treat the case when there is a nonzero value for the angular separation, $\vec{\vartheta}$. The section after that will apply these statistical results to the task of developing expressions for adaptive optics imaging systems and laser transmitters with and without random apodization correction. The final sections will then be concerned with the development of numerical results and conveniently usable approximations, and with a discussion of these results.

1.2 Ideal System Performance

In this section we shall consider the performance of an ideal system (imaging or laser transmitter) in two senses of the word ideal. In the first sense we shall take ideal to mean that there are no turbulence effects along the propagation path, so that system performance is truly diffraction limited. This will be the case treated in subsection 1.2.1 . Our objective there will be to present some basic formulations and to develop some reference results against which other results can be compared. Along the way some of the fundamental notation and assumptions will be introduced.

In its second sense of the term "ideal system" we shall treat the term as meaning that there is no anisoplanatism problem, i.e., that $\vec{\vartheta}$ equals zero. This will be treated in subsection 1.2.2 . Our objective there will be to explore in a limited way the implications of the random apodization. It is in this subsection that we will develop an expression for the effect of random apodization, i.e., for beacon power density variations, when there is no provision in the adaptive optics for correcting for turbulence induced random apodization. This subsection will also consider the question of developing an appropriate control law for adaptive optics compensation of the random apodization. It is in this subsection that it will be shown that the control law for random apodization correction is fundamentally different for an imaging system than it is for a laser transmitter, and that while the imaging system's performance can be made to be exactly equal to the diffraction limited value, this is not the case for the laser transmitter.

1.2.1 Performance in the Absence of Turbulence Effects

In this subsection we shall consider the performance of an imaging system and of a laser transmitter in the absence of any turbulence effects, each assumed to be ideally fabricated and aligned. For the imaging system we shall take the optical transfer function (OTF) of the system, as a function of spatial frequency to be our measure of system performance. For the laser transmitter we shall take the antenna gain, as our measure of system performance.

Here, and throughout this chapter we shall assume that the relevant wavelength is λ. This will be the same wavelength for the reference beacon, the laser transmitter, and the imaging system. Likewise, throughout this chapter we shall assume that our optical system has a circular unobscured aperture of diameter D. Accordingly, taking $\vec{r}$ to denote a two-dimensional position vector on a plane containing the aperture with the origin corresponding to the center of the aperture, we can define the aperture by the function $W(\vec{r})$, where

$$W(\vec{r}) = \begin{cases} 1, & \text{if } |\vec{r}| \le D/2 \\ 0, & \text{if } |\vec{r}| > D/2 \end{cases} \quad . \tag{2.1}$$

We shall consider a reference beacon located at a range R from the (center of the) aperture, i. e., from the origin and displaced from the z-axis direction by an angle $-\frac{1}{2}\vec{\vartheta}$, and a laser aim-point or a point source to be imaged located at the same range, R, but displaced from the z-axis direction by an angle $+\frac{1}{2}\vec{\vartheta}$. Thus the angular separation between the point source to be imaged or the laser aim-point and the reference beacon is equal to $\vec{\vartheta}$. The z-axis is nominally perpendicular to the aperture plane. Based on the assumption that $|\vec{\vartheta}|$ is very small, then the z-axis defines the nominal propagation direction.

The path length from a point $\vec{r}$ in the aperture plane to the beacon is $L(\vec{r}, -\frac{1}{2}\vec{\vartheta})$, while that to the point source or aim-point is $L(\vec{r}, \frac{1}{2}\vec{\vartheta})$, where

$$L(\vec{r}, \vec{\theta}) = (R^2 + |\vec{r} - \vec{\theta} R|^2)^{1/2} \quad . \tag{2.2}$$

Based on the assumption that $|\vec{r}|$ is much smaller than R, as well as that $|\vec{\theta}|$ is very small we can rewrite Eq. (2) in the approximate form

$$L(\vec{r}, \vec{\theta}) \approx R + \frac{r^2}{2R} - \vec{r} \cdot \vec{\theta} + \tfrac{1}{2} R \theta^2 \quad . \tag{2.3}$$

To provide focusing at range R the optical power elements of the optical system must produce an $\vec{r}$ - dependent optical path length $O(\vec{r})$, where

$$O(\vec{r}) = -\tfrac{1}{2} r^2/R \quad . \tag{2.4}$$

With these notational matters taken care of we are now ready to consider the development of expressions for the diffraction limited OTF and antenna gain. We shall take up the OTF problem first.

1.2.1.1 OTF in the Absence of Turbulence Effects

In this section we shall fairly closely follow the imaging system analysis presented in one of our earlier works[3]. We start by noting that if the wavefunction from a point source [at $(R, \vec{\theta})$] after passing through the optics (and adaptive optics, if any) can be written as the pupil function $U(\vec{r})$, then for an optical system with focal length F the wavefunction, $u(\vec{x})$, at position $\vec{x}$ in the focal plane of our imaging system, where $\vec{x}$ is a two-dimensional position vector on the focal plane, can be written as

$$u(\vec{x}) = \frac{i}{\lambda F} \int d\vec{r}\; U(\vec{r}) \exp(-ik\vec{r} \cdot \vec{x}/F) \quad , \tag{2.5}$$

where here and throughout this chapter

$$k = 2\pi/\lambda \quad . \tag{2.6}$$

The power density in the focal plane associated with this wavefunction can be written as

$$\begin{aligned} \mathcal{P}(\vec{x}) &= \tfrac{1}{2} |u(\vec{x})|^2 \\ &= \tfrac{1}{2} u^*(\vec{x})\; u(\vec{x}) \quad . \end{aligned} \tag{2.7}$$

Our interest is in the OTF, which can be evaluated as the Fourier transform of the power density, $\mathcal{P}(\vec{x})$, since $\mathcal{P}(\vec{x})$ represents the image of a point source. Accordingly we can write for the optical transfer function for

spatial frequency $\vec{f}$ (with $\vec{f}$ measured in cycles per radian-field-of-view)

$$\mathfrak{T}(\vec{f}) = B \int d\vec{x}\, \mathscr{P}(\vec{x}) \exp(-2\pi i \vec{f}\cdot\vec{x}/F) \quad , \qquad (2.8)$$

where B is a normalization constant choosen to make the OTF of zero spatial frequency equal to unity, i. e., so that

$$\mathfrak{T}(0) \equiv 1 \quad , \qquad \text{(normalization condition)} . \qquad (2.9)$$

When we substitute Eq. (5) into Eq. (7) twice, once with the variable of integration denoted by $\vec{r}$ and once by $\vec{r}\,'$, and then substitute that result into Eq. (8), we can after making the product of integrals into a double integral, write the result as

$$\mathfrak{T}(\vec{f}) = \tfrac{1}{2} B \lambda^{-2} F^{-2} \iint d\vec{x}\, d\vec{r}\, d\vec{r}\,' U^{*}(\vec{r}\,') U(\vec{r})$$

$$\times \exp\{ i(k/F)\vec{x}\cdot[\vec{r}\,' - (\vec{r} + \lambda\vec{f})]\} \quad . \qquad (2.10)$$

We note that the combination of the $\vec{x}$-integration and the $\vec{r}$-integration in Eq. (10) can be considered to be a repeated Fourier integral. As is well known[4], in accordance with the equation

$$\iint d\vec{p}\,' d\vec{g}\, g(\vec{p}\,') \exp[\pm i\vec{g}\cdot(\vec{p}\,' - \vec{p})] = (2\pi)^n g(\vec{p}) , \qquad (2.11)$$

where $g(\vec{p})$ is any reasonably well behaved function of $\vec{p}$, and $\vec{p}$, $\vec{p}\,'$, and $\vec{g}$ are n-dimensional vectors, the repeated Fourier integral recovers the starting function. Accordingly we can rewrite Eq. (10) as

$$\mathfrak{T}(\vec{f}) = \tfrac{1}{2} B \lambda^{-2} F^{-2} (F/k)^2 (2\pi)^2 \int d\vec{r}\, U^{*}(\vec{r} + \lambda\vec{f}) U(\vec{r})$$

$$= \tfrac{1}{2} B \int d\vec{r}\, U^{*}(\vec{r} + \lambda\vec{f}) U(r) \quad , \qquad (2.12)$$

reducing our result to a simple integral over the aperture.

To evaluate the normalization constant, B, we make use of Eq's. (9), and (12). This allow us to write

$$B = \{ \int d\vec{r}\, [\tfrac{1}{2} U^{*}(\vec{r}) U(\vec{r})] \}^{-1} \quad . \qquad (2.13)$$

Further simplification of this result is generally possible based on the fact that in this case the power density in the aperture, $\frac{1}{2} U^*(\vec{r}) U(\vec{r})$, and in the more general case the average power density in the aperture, $\langle \frac{1}{2} U^*(\vec{r}) U(\vec{r}) \rangle$, is not a function of aperture position, $\vec{r}$. Thus we can write for the power density in the aperture

$$\mathscr{P}_A = \tfrac{1}{2} U^*(\vec{r}) U(\vec{r}) \quad , \qquad (2.14)$$

or in the more general case

$$\mathscr{P}_A = \langle \tfrac{1}{2} U^*(\vec{r}) U(\vec{r}) \rangle \quad , \qquad (2.14')$$

where the angle brackets, $\langle \ldots \rangle$, denote an ensemble average. Making use of Eq.'s (1) and (14), Eq. (13) can be reduced to the form

$$\begin{aligned} B &= [\mathscr{P}_A \int d\vec{r}\; W(\vec{r})\,]^{-1} \\ &= (\tfrac{1}{4}\pi D^2 \mathscr{P}_A)^{-1} \quad . \qquad (2.15) \end{aligned}$$

For the diffraction limited case which we are concerned with here the pupil function $U(\vec{r})$ is defined by an amplitude, A, which is not a function of position, $\vec{r}$, a path length induced phase shift, L, and a focusing phase shift, O. Thus we can write for a source at $(R, \vec{\theta})$

$$U(\vec{r}) = A\; W(\vec{r}) \exp\{ik[L(\vec{r}, \vec{\theta}) + O(\vec{r})]\} , \qquad (2.16)$$

where we recall that L, O, and k are as defined by Eq.'s (2) or (3), (4), and (6), respectively. We can rewrite this as

$$U(\vec{r}) = A\; W(\vec{r}) \exp[ik(R + \tfrac{1}{2} R\theta^2 - \vec{r}\cdot\vec{\theta})] , \qquad (2.17)$$

from which in conjunction with Eq. (14), it follows that

$$\mathscr{P}_A = \tfrac{1}{2} A^* A\; W(\vec{r}) \quad , \qquad (2.18)$$

and

$$B = (\tfrac{1}{8}\pi D^2 A^* A)^{-1} \quad . \qquad (2.19)$$

When we substitute Eq.'s (17) and (19) into Eq. (12) and carry out the obvious simplifications we get the result that

$$\mathfrak{T}(\vec{f}) = (\tfrac{1}{4}\pi D^2)^{-1} \int d\vec{r}\ W(\vec{r})\, W(\vec{r}+\lambda\vec{f})\ \exp(ik\lambda\vec{f}\cdot\vec{\theta})$$

$$\Rightarrow (\tfrac{1}{4}\pi D^2)^{-1} \int d\vec{r}\ W(\vec{r})\, W(\vec{r}+\lambda\vec{f}) \quad . \qquad (2.20)$$

In writing Eq. (20) we have dropped the factor $\exp(ik\lambda\vec{f}\cdot\vec{\theta})$ as this is just a phase shift factor indicative of the fact that the point-source is not on the z-axis but displaced from it by an angle $\vec{\theta}$. The modulus of this factor is unity and for the on-axis point-source the factor would be exactly unity. Thus in dropping this factor we, in essence, make the point-source define the on-axis direction for OTF purposes.

The integral in Eq. (20) represents the area of overlap of two circles of diameter D with centers displaced by amount $\lambda\vec{f}$. It is easy to show from some simple trigonometric analysis that if two circles of diameter D have their centers separated by a distance ρ, then the area of overlap of the circles is $\frac{1}{4}\pi D^2 K(\rho)$, where

$$K(\rho) = \begin{cases} \frac{2}{\pi}\left\{ \cos^{-1}(\rho/D) - (\rho/D)\left[1-(\rho/D)^2\right]^{1/2} \right\}, & \text{if } \rho \le D \\ 0, & \text{if } \rho > D \end{cases} \quad . \qquad (2.21)$$

Thus we can rewrite Eq. (20) as

$$\mathfrak{T}_{DL}(\vec{f}) = K(\lambda f)$$

$$= \begin{cases} \frac{2}{\pi}\ \cos^{-1}(\lambda f/D) - (\lambda f/D)\left[1-(\lambda f/D)^2\right]^{1/2} \}, & \text{if } \lambda f \le D \\ 0, & \text{if } \lambda f > D \end{cases} \quad . \qquad (2.22)$$

The subscript "DL" is added here to make explicit the fact that this result applies for the case of diffraction limited operation. Eq. (22) represents our basic result for the OTF of an ideal imaging system in the absence of turbulence. In the next subsection we shall consider the corresponding problem for a laser transmitter.

1.2.1.2 Antenna Gain in the Absence of Turbulence

The antenna gain of a laser transmitter, G, is most usefully defined as the ratio of the laser powered density at the aim-point, $\mathcal{P}_{AP}$, divided by the total laser power transmitted, P_T, and scaled for the range,

R , from the laser transmitter to the aim-point. Thus we would write

$$G = (\mathscr{P}_{AP}/P_T)\, R^2 \qquad . \qquad (2.23)$$

Expressed in this form the antenna gain has the dimension of inverse steradians. It is in effect a measure of the (inverse of the) beam spread of the laser transmitter.*

If the laser wavefunction leaving the aperture is $U(\vec{r})$, then the wavefunction at an aim-point at $(R, \vec{\theta})$ can be written as

$$u_{AP} = -\frac{i}{\lambda R} \int d\vec{r}\; U(\vec{r}) \exp[ikL(\vec{r} \;\; \vec{\theta})], \qquad , \qquad (2.24)$$

where L is, as defined by Eq.'s (2) or (3), the path length between the point at $\vec{r}$ on the aperture plane and the aim-point. To focus the laser beam on the aim-point the laser transmitter optics would cause the wave-function leaving the aperture to have a form expressable as

$$U(\vec{r}) = A\; W(\vec{r}) \;\exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}\,]\} \qquad , \qquad (2.25)$$

where the $\vec{r}\cdot\vec{\theta}$ - term in the exponential-function indicates that the laser beam is directed at the aim-point, and the $O(\vec{r})$-term indicates that the laser beam is focused at the range, R , of the aim-point. Making use of Eq.'s (3), (4), and (25), Eq. (24) can be reduced to the form

$$u_{AP} = -\frac{i}{\lambda R} \exp[ikR(1+\tfrac{1}{2}\theta^2)]\; A \int d\vec{r}\;\; W(\vec{r}) \qquad . \qquad (2.26)$$

Taking note of Eq. (1) we can further reduce this to

$$u_{AP} = -\frac{i}{\lambda R} \exp[ikR(1+\tfrac{1}{2}\theta^2)]\; A\; \tfrac{1}{4}\pi D^2 \qquad . \qquad (2.27)$$

The laser power density at the aim-point can be written as

$$\mathscr{P}_{AP} = \tfrac{1}{2}\, u_{AP}^{*}\; u_{AP} \qquad , \qquad (2.28)$$

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* The antenna gain as we have defined it can be related to the more conventional antenna-gain-relative-to-isotropic, by multiplying our value by 4π.

which can be reduced, by means of Eq. (27) to the form

$$\mathcal{P}_{AP} = \frac{1}{2} \lambda^{-2} R^{-2} A^{*} A \left(\frac{1}{4}\pi D^{2}\right)^{2} \quad . \tag{2.29}$$

The laser power density at the aperture is as defined by Eq. (14) and in view of Eq.'s (1) and (25) can be written as

$$\mathcal{P}_{A} = \frac{1}{2} A^{*} A \; W(\vec{r}) \quad . \tag{2.30}$$

The total laser power transmitted is of course just

$$\begin{aligned} P_{T} &= \int d\vec{r} \; \mathcal{P}_{A} \\ &= \frac{1}{2} A^{*} A \int d\vec{r} \; W(\vec{r}) \\ &= \frac{1}{8} \pi D^{2} A^{*} A \quad . \end{aligned} \tag{2.31}$$

When we substitute Eq.'s (29) and (31) into Eq. (23) we obtain for the antenna gain, the result that

$$G_{DL} = \frac{1}{4}\pi (D/\lambda)^{2} \quad . \tag{2.32}$$

Here again the subscript "DL" has been added to make explicit the fact that this result applies for the case of diffraction limited operation.

With these results in hand, i.e., Eq.'s (22) and (32) we have completed the desired analysis of system performance in the absence of turbulence. In the next subsection we shall turn our attention to the other sense in which we used the term "ideal system", namely an adaptive optics system in which there is no angular separation between the directions from the transmitter to the point-source or aim-point and to the reference beacon.

1.2.2 Adaptive Optics Performance With Turbulence But No Angular Separation

To make our notation sufficiently general, so that it can be used in the latter parts of this chapter, we shall consider a reference beacon located at $(R, \vec{\theta}_B)$ and a target (either a point-source to be imaged or a laser aim-point) located at $(R, \vec{\theta}_T)$. In this section we will, at an appropriate point introduce the fact that $\vec{\theta}_B = 0$ and that $\vec{\theta}_T = 0$, but for much of the deviation we shall ignore these facts so as to keep the results generally applicable. We shall

consider an adaptive optics system which can perfectly sense the wavefunction perturbation induced on the beacon signal by propagation through turbulence, and which can apply exactly the desired correction. With no angular separation between the directions there is no reason why the system performance should not be ideal — except for the matter of how to handle power density variations across the system's aperture. And it is indeed just this matter that is the central concern of this subsection.

Before we start a detailed examination of this matter it is necessary that at this point we introduce the appropriate notation for the propagation statistics. For a point source located at position $\vec{r}$ on the aperture and traveling to an "end-point" position defined by the range R and angular deviation from the z-axis of $\vec{\theta}$, i.e., by $(R, \vec{\theta})$, if the wavefunction at the end-point should be $u_0(\vec{r}, \vec{\theta})$ if there were no turbulence along the propagation path, and if the instantaneous random value of the wavefunction is $u(\vec{r}, \vec{\theta})$, it is convenient to make manifest the turbulence effects by writing

$$u(\vec{r}, \vec{\theta}) = u_0(\vec{r}, \vec{\theta}) \exp[i\,\psi(\vec{r}, \vec{\theta})] \quad . \tag{2.33}$$

The quantity $\psi(\vec{r}, \vec{\theta})$, which we call the "complex-phase perturbation" serves as a measure of the effects of turbulence. It is convenient to write $\psi(\vec{r}, \vec{\theta})$ in terms of its real and imaginary parts, $\phi(\vec{r}, \vec{\theta})$ and $-\ell(\vec{r}, \vec{\theta})$ respectively. Thus we have

$$\psi(\vec{r}, \vec{\theta}) = \phi(\vec{r}, \vec{\theta}) - \ell(\vec{r}, \vec{\theta}) \quad . \tag{2.34}$$

The quantity $\phi(\vec{r}, \vec{\theta})$ is called the "phase perturbation" or the "real phase perturbation". It is generally what is intended when one speaks of "correcting the phase" of a turbulence distorted wavefunction. The quantity $\ell(\vec{r}, \vec{\theta})$ is generally called the "log-amplitude perturbation". It is a (logarithmic) measure of power density variations.

By virtue of the reciprocity theorem[2] the complex phase perturbation $\psi(\vec{r}, \vec{\theta})$, is not only a measure of the turbulence induced perturbation of a point source traveling from position $\vec{r}$ on the aperture plane to a target position at $(R, \vec{\theta})$, but serves equally well as a measure of the turbulence induced point source at the target position $(R, \vec{\theta})$ propagating back to the position $\vec{r}$ on the aperture plane. As a consequence, $\ell(\vec{r}, \vec{\theta}_B)$ is a measure of the power density variation of the beacon signal at the aperture.

The key to our analysis in this subsection lies in our ability to form the ensemble average of the exponential function of some linear combination of complex phase terms. The key to being able to do this lies in the fact that if χ is a gaussian random variable, or linear combination of several gaussian random variables, then it is easy to show by carrying out an integration over the probability density, that

$$\langle \exp(\alpha\chi) \rangle = \exp(\alpha\overline{\chi}) \exp[\tfrac{1}{2}\alpha^2 \langle (\chi - \overline{\chi})^2 \rangle] \quad , \qquad (2.35)$$

where α is a constant, and

$$\overline{\chi} = \langle \chi \rangle \quad , \qquad (2.36)$$

is the mean value of our random variable(s).

Use of Eq. (35) allows proof, by means of conservation of energy arguments, that for the log-amplitude pertubation, ℓ,

$$\overline{\ell} = \langle (\ell - \overline{\ell})^2 \rangle \quad , \qquad (2.37)$$

where we have written ℓ in place of $\ell(\vec{r}, \vec{\theta})$ simply as a matter of convenience. With these results in hand we are now ready to take up the first of the several cases to be considered in this subsection, namely the ensemble average OTF of an adaptive optics imaging system when there is no separation between the point-source (target) direction, $\vec{\theta}_T$, and the reference beacon direction, $\vec{\theta}_B$, for the case of adaptive optics that only corrects for the real phase perturbations, $\phi(\vec{r}, \vec{\theta})$, and not for the power density or log-amplitude perturbations, $\ell(\vec{r}, \vec{\theta})$, i. e., not for random apodization.

1.2.2.1 <u>Ideal Adaptive Optics Imaging But With No Random Apodization Correction</u>

Our starting point for the OTF analysis to be performed here consists of Eq.'s (12) and (13) in ensemble average form. The ensemble average OTF (which can be shown to correspond both to the ensemble average of the instantaneous random OTF, and to the OTF associated with an ensemble average of the instantaneous random image) can be written as

$$\langle \mathcal{I}(\vec{f}) \rangle = \tfrac{1}{2} \langle B \rangle \int d\vec{r} \, \langle U^{*}(\vec{r} + \lambda \vec{f}) \, U(\vec{r}) \rangle \quad , \tag{2.38}$$

with the ensemble average normalization constant written as

$$\langle B \rangle = \{ \int d\vec{r} \, \tfrac{1}{2} \langle U^{*}(\vec{r}) \, U(\vec{r}) \rangle \}^{-1} \quad , \tag{2.39}$$

It is to be noted that $\langle B \rangle$ is not the ensemble average of the instantaneous random values of the normalization constant B , and thus is not obtained directly from Eq. (13) by applying the ensemble average brackets, $\langle \ldots \rangle$ to both sides of Eq. (13). Rather the value of $\langle B \rangle$ is obtained from Eq. (38) in just the same way that Eq. (13) is obtained from Eq. (12), i. e., by reference to Eq. (9), the relevant form at which we write here as

$$\langle \mathcal{I}(0) \rangle \equiv 1 \quad . \tag{2.40}$$

Eq. 's (38) and (30) would be relevant to any imaging system — its just a matter of the nature of the specifications of the pupil function, $U(\vec{r})$. To make it explicit here that we are dealing with an adaptive optics imaging system without any random apodization correction we shall use the subscript IW/O. Thus we would write as the locally relevant forms of Eq. 's (38) and (39)

$$\langle \mathcal{I}_{IW/O}(\vec{f}) \rangle = \tfrac{1}{2} \langle B_{IW/O} \rangle \int d\vec{r} \, \langle U^{*}_{IW/O}(\vec{r} + \lambda \vec{f}) \, U_{IW/O}(\vec{r}) \rangle , \tag{2.41}$$

$$\langle B_{IW/O} \rangle = \{ \int d\vec{r} \, \tfrac{1}{2} \langle U^{*}_{IW/O}(\vec{r}) \, U_{IW/O}(\vec{r}) \rangle \}^{-1} \quad . \tag{2.42}$$

The pupil function, $U_{IW/O}(\vec{r})$, is in part as given by Eq. (16) , but in addition there is an exponential function of the complex phase, $\psi(\vec{r}, \vec{\theta}_T)$, associated with propagation through turbulence from the (target) point-source to be imaged at $\vec{\theta}_T$, less the turbulence induced real phase $\phi(\vec{r}, \vec{\theta}_B)$, sensed in viewing the reference beacon at $\vec{\theta}_B$. Thus we can write

$$U_{IW/O}(\vec{r}) = A \; W(\vec{r}) \exp \{ ik [L(\vec{r}, \vec{\theta}_T) + O(\vec{r})] \\ + i [\psi(\vec{r}, \vec{\theta}_T) - \phi(\vec{r}, \vec{\theta}_B)] \} \quad . \tag{2.43}$$

Making use of Eq.'s (3), (4), and (34) this can be rewritten as

$$U_{I\,W/O}(\vec{r}) = A\,W(\vec{r})\exp\{ik[R(1+\tfrac{1}{2}\dot{\vec{\theta}}_T) + O(\vec{r})] + i[\psi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)]\} \quad . \qquad (2.44)$$

When we twice substitute this expression into Eq. (42) and appropriately simplify the results we get

$$\langle B_{I\,W/O}\rangle = \{\tfrac{1}{2}A^*A\int d\vec{r}\,W(\vec{r})\langle\exp[2\ell(\vec{r},\vec{\theta}_T)]\rangle\}^{-1} \quad . \qquad (2.45)$$

Stationarity of the statistics of the real and imaginary parts of the complex phase insures that neither the mean value of the log-amplitude

$$\bar{\ell} = \langle\ell(\vec{r},\vec{\theta})\rangle \quad , \qquad (2.46)$$

nor the covariance of the log-amplitude

$$C_\ell(\vec{\rho},\vec{\vartheta}) = \langle[\ell(\vec{r}+\vec{\rho},\vec{\theta}+\vec{\vartheta}) - \bar{\ell}][\ell(\vec{r},\vec{\theta}) - \bar{\ell}]\rangle \quad , \qquad (2.47)$$

is a function $\vec{r}$ or $\vec{\vartheta}$. In accordance with Eq. (35) we can thus rewrite Eq. (45) as

$$\langle B_{I\,W/O}\rangle = \{\tfrac{1}{2}A^*A\exp(2\bar{\ell})\exp[2C_\ell(0,0)]\int d\vec{r}\,W(\vec{r})\}^{-1}. \quad (2.48)$$

By virtue of Eq. (37) the log-amplitude mean value, $\bar{\ell}$, and variance, $C_\ell(0,0)$, exactly cancel. [This exact cancellation is directly associated with the fact of energy conservation — which is what gave rise to Eq. (37) in the first place.] Thus we can write

$$\langle B_{I\,W/O}\rangle = \{\tfrac{1}{2}A^*A\int d\vec{r}\,W(\vec{r})\}^{-1} \quad , \qquad (2.49($$

which in view of Eq. (1) can be reduced to the form

$$\langle B_{I\,W/O}\rangle = (\tfrac{1}{8}\pi D^2 A^*A)^{-1} \quad . \qquad (2.50)$$

This result, which is identical to Eq. (19) for the no turbulence case, applies whether or not there is any angular separation between the target point-source direction, $\vec{\theta}_T$, and the reference beacon direction, $\vec{\theta}_B$. For the balance of this section we shall, however, restrict our attention to the more limited (ideal) case which is the subject of this subsection, where $\vec{\theta}_T$ and $\vec{\theta}_B$ are both equal to zero, so that both target point-source and

reference beacon lie on the z-axis.

With this restriction Eq. (44) reduces to

$$U_{1W/0}(\vec{r}) = A\,W(\vec{r})\exp[ikR + \ell(\vec{r},0)] \quad . \tag{2.51}$$

From this it follows that

$$\langle U^*_{1W/0}(\vec{r}+\vec{\rho})\,U_{1W/0}(\vec{r})\rangle = A^* A W(\vec{r}+\vec{\rho})\,W(\vec{r}) \times \langle \exp[\ell(\vec{r}+\vec{\rho},0) + \ell(\vec{r},0)]\rangle \quad . \tag{2.52}$$

Making use of Eq. (52) to allow evaluation of the ensemble average of an exponential function, we can easily show that

$$\langle \exp[\ell(\vec{r}+\vec{\rho},0) + \ell(\vec{r},0)]\rangle = \exp(2\,\bar{\ell}) \times \exp[C_\ell(0,0) + C_\ell(\vec{\rho},0)] \quad . \tag{2.53}$$

From Eq. (37) it then follows that

$$\langle \exp[\ell(\vec{r}+\vec{\rho},0) + \ell(\vec{r},0)]\rangle = \exp[-C_\ell(0,0) + C_\ell(\vec{\rho},0)] . \tag{2.54}$$

When we substitute this expression into Eq. (53) we get the result that

$$\langle U^*_{1W/0}(\vec{r}+\vec{\rho})\,U_{1W/0}(\vec{r})\rangle = A^* A W(\vec{r}+\vec{\rho})\,W(\vec{r}) \times \exp[-C_\ell(0,0) + C_\ell(\vec{\rho},0)] . \tag{2.55}$$

By means of this result we can now reduce Eq. (41) to the form

$$\mathcal{I}_{1W/0}(\vec{f}) = \tfrac{1}{2}A^* A \exp[-C_\ell(0,0) + C_\ell(\lambda\vec{f},0)]\,\langle B_{1W/0}\rangle \times \int d\vec{r}\,W(\vec{r}+\lambda\vec{f})\,W(\vec{r}) \quad . \tag{2.56}$$

This integral over $\vec{r}$ is the one discussed just before Eq. (22). Making use of this fact we can write

$$\mathcal{I}_{1W/0}(\vec{f}) = \tfrac{1}{2}A^* A \exp[-C_\ell(0,0) + C_\ell(\lambda\vec{f},0)]\,\langle B_{1W/0}\rangle \times \tfrac{1}{4}\pi D^2 K(\lambda f) \quad , \tag{2.57}$$

which, in view of Eq.'s (22) and (50) reduces to

$$\mathfrak{T}_{IW/O}(\vec{f}) = \exp[-C_\ell(0,0) + C_\ell(\lambda\vec{f},0)]\,K(\lambda f)$$

$$= \exp[-C_\ell(0,0) + C_\ell(\lambda\vec{f},0)]\,\mathfrak{T}_{DL}(\vec{f}) \quad . \tag{2.58}$$

The key thing to note in interpreting this result is that for very low spatial frequencies $C_\ell(\lambda\vec{f},0)$ is very nearly equal to $C_\ell(0,0)$, so that the adaptive optics imaging system without random apodization correction provides essentially the diffraction limited OTF, $\mathfrak{T}_{DL}(\vec{f})$ — while for the higher spatial frequencies, which are what are really of interest to us, $C_\ell(\lambda\vec{f},0)$ is very nearly equal to zero, so that the OTF is less than diffraction limited by a factor of $\exp[-C_\ell(0,0)]$. If the log-amplitude variance is small enough this factor is very close to unity and there would seem to be no real need for a random apodization correction capability in the imaging system's adaptive optics. If, however, the log-amplitude variance, $C_\ell(0,\vec{0})$, is not particularly small then the exponential factor can result in a nontrivial reduction factor — so that it might be useful to be able to have the adaptive optics compensate for random apodization. We take up an analysis of this case next.

1.2.2.2 Ideal Adaptive Optics Imaging With Random Apodization Correction

Using the subscript IW to denote the fact that the expression relates to an adaptive optics imaging system with compensation of random apodization, we would write in place of Eq.'s (41) and (42), the equations

$$\langle \mathfrak{T}_{IW}(\vec{f})\rangle = \tfrac{1}{2}\langle B_{IW}\rangle \int d\vec{r}\ \langle U^*_{IW}(\vec{r}+\lambda\vec{f})\,U_{IW}(\vec{r})\rangle\,, \tag{2.59}$$

for the ensemble average OTF, and

$$\langle B_{IW}\rangle = \{\int d\vec{r}\ \tfrac{1}{2}\langle U^*_{IW}(\vec{r})\,U_{IW}(\vec{r})\rangle\}^{-1} \quad , \tag{2.60}$$

for the corresponding normalization constant.

For adaptive optics imaging when there is no angular separation between the target point-source to be imaged and the reference beacon, it is obvious that there will be no residual turbulence effects if the adaptive optics random apodization correction makes the system's transmission

inversely proportional to the power density of the beacon reference signal over the aperture*. We shall shortly see from our equations that this is exactly the case. We start by noting that this random apodization scheme would come about if the adaptive optics imposed a complex phase correction in which the imaginary part were the negative of the imaginary part of the complex phase associated with the beacon. Thus, in place of Eq. (43) we would have

$$U_{tw}(\vec{r}) = A\, W(\vec{r}) \exp\{ik[L(\vec{r},\vec{\theta}_T) + O(\vec{r})] + i[\phi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)] - \ell(\vec{r},\vec{\theta}_B)\} \quad , \qquad (2.61)$$

and in view of Eq. (34), and using Eq.'s (3) and (4) this can be rewritten as

$$U_{tw}(\vec{r}) = A\, W(\vec{r}) \exp\{ik[R(1+\tfrac{1}{2}\theta_T^2) - \vec{r}\cdot\vec{\theta}_T] + i[\phi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)] + [\ell(\vec{r},\vec{\theta}_T) - \ell(\vec{r},\vec{\theta}_B)]\}. \qquad (2.62)$$

For the ideal case of interest to us here, in which there is no angular separation between the target point-source to be imaged and the beacon reference, with both lying on the z-axis so that $\vec{\theta}_T$ equals $\vec{\theta}_B$, then obviously the two complex phase function in Eq. (62) will cancel and we get

$$U_{tw}(\vec{r}) = A\, W(\vec{r}) \exp\{ik[R(1+\tfrac{1}{2}\theta_T^2) - \vec{r}\cdot\vec{\theta}_T]\} \quad . \qquad (2.63)$$

This expression is manifestly independent of any turbulence effects, and by comparing of it with Eq. (16) we can see that it will lead to the conclusion that the OTF will be diffraction limited. If we carried out the same

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* It is not entirely clear exactly how the random apodization correction would be affected. Transmission reduction is no special problem, but where the beacon power density is very large the transmission would have to be made correspondingly large. Leaving out the concept of laser amplifiers (and nearly monochromatic imaging) it would seem that affecting random apodization correction would require acceptance of a very low average transmission, so that increases as well as decreases in transmission would be possible and we would have to accept a limited dynamic range for the compensation. In the analysis we have considered this limited range, as well as the reduced average transmission as things we could ignore.

calculations as in subsection 1.2.1.1 we would get exactly the same results, which we can write as

$$\langle \mathfrak{T}_{IW}(\vec{f}) \rangle = \mathfrak{T}_{DL}(\vec{f}) \qquad . \qquad (2.64)$$

This very nice result would, of course, not be expected to hold if there were a nonzero angular separation, $\vec{\vartheta}$, between the target direction and the beacon direction — but this is a matter that we shall take up later. In the next two subsections we shall turn our attention to the adaptive optics laser transmitter problem in the presence of turbulence effects, but for the ideal conditions case of no angular separation between the directions to the target aim-point and to the reference beacon.

1.2.2.3 Ideal Laser Transmitter Adaptive Optics But With No Random Apodization Correction

In exactly the same way that we arrived at Eq. (23) as the appropriate expression for the non-random (or instantaneous) laser transmitter antenna gain, we would write for the ensemble average antenna gain

$$\langle G \rangle = (\langle \mathcal{P} \rangle / \langle P \rangle)\, R^2 \qquad , \qquad (2.65)$$

where $\langle \mathcal{P} \rangle$ denotes the ensemble average power density at the aim-point and $\langle P \rangle$ denotes the ensemble average laser transmitter power. It is significant to note that the ensemble average antenna gain, $\langle G \rangle$, is in general not the ensemble average of the instantaneous random antenna gain, i.e., Eq. (65) is not obtained by simply applying ensemble averaging to both sides of Eq. (23). Rather, Eq. (65) is presented on the same basis as was Eq. (23), only this time as the ratio of <u>average</u> power density and <u>average</u> transmitted power.

The ensemble average laser power density at the aim-point, $\langle \mathcal{P} \rangle$, can be written as

$$\langle \mathcal{P} \rangle = \frac{1}{2} \langle u^* u \rangle \qquad , \qquad (2.66)$$

which can be directly inferred from Eq. (28). Here u is the laser transmitter produced random wavefunction at the aim-point. This wavefunction can be written in terms of the laser transmitter's random pupil function since

it incorporates the effects of the adaptive optics corrections, which are random in the sense that they follow turbulence effects. In place of Eq. (24) we now have the expression

$$u = -\frac{i}{\lambda R}\int d\vec{r}\, U(\vec{r}) \;\exp\{i[k L(\vec{r},\vec{\theta}_T) + \psi(\vec{r},\vec{\theta}_T)]\} . \qquad (2.67)$$

Thus far over notation has been quite general, not indicating whether we were allowing for random apodization correction. To write down an expression for the pupil function we must be explicit on that matter. Using the subscript LT W/O to denote an adaptive optics laser transmitter without any random apodization correction we would write, as an extension of Eq. (25)

$$U_{LTW/O}(\vec{r}) = A\, W(\vec{r}) \exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}_T] - i\phi(\vec{r},\vec{\theta}_B)\}, \qquad (2.68)$$

as the adaptive optics corrected pupil function. With the same subscript notation Eq.'s (65), (66), and (67) would be rewritten as

$$\langle G_{LTW/O}\rangle = (\langle \mathcal{P}_{LTW/O}\rangle / \langle P_{LTW/O}\rangle)\, R^2 \quad , \qquad (2.69)$$

$$\langle \mathcal{P}_{LTW/O}\rangle = \tfrac{1}{2}\langle u^*_{LTW/O}\; u_{LTW/O}\rangle \quad , \qquad (2.70)$$

and

$$u_{LTW/O} = -\frac{i}{\lambda R}\int d\vec{r}\; U_{LTW/O}(\vec{r}) \exp\{i[k L(\vec{r},\vec{\theta}_T) + \psi(\vec{r},\vec{\theta}_T)]\}. \qquad (2.71)$$

Making use of Eq.'s (3), (4), (34), (67), and (68) we can rewrite Eq. (71) as

$$u_{LTW/O} = -\frac{i}{\lambda R}\exp[ikR(1+\tfrac{1}{2}\vec{\theta}_T^{\,2})]\, A \int d\vec{r}\; W(\vec{r})$$

$$\times \exp\{i[\phi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)] + \ell(\vec{r},\vec{\theta}_T)\} . \qquad (2.72)$$

We note that the ensemble average transmitter laser power can be written as

$$\langle P_{LTW/O}\rangle = \int d\vec{r}\; \tfrac{1}{2}\langle U^*_{LTW/O}(\vec{r})\; U_{LTW/O}(\vec{r})\rangle \quad , \qquad (2.73)$$

and that from Eq. (68) it then follows that

$$\langle P_{LTW/O}\rangle = \int d\vec{r}\; \tfrac{1}{2} A^* A\; W(\vec{r})$$

$$= \tfrac{1}{8}\,\pi D^2 A^* A \quad . \qquad (2.74)$$

The results up to this point have been developed without making any use of the fact that the target aim-point and the reference beacon both lie on the

z-axis, so that $\vec{\theta}_T$ and $\vec{\theta}_B$ are both equal to zero and there is no angular separation between the directions to the target aim-point and to the reference beacon.

When we make use of the zero values of $\vec{\theta}_T$ and $\vec{\theta}_B$, Eq. (72) reduces to

$$u_{LTW/O} = -\frac{i}{\lambda R} \exp(ikR)\, A \int d\vec{r}\; W(\vec{r}) \exp[\ell(\vec{r},0)] \;. \qquad (2.75)$$

We now substitute Eq. (75) twice into Eq. (70), make a double integral out of the product of integrals, and interchange the order of integrations and ensemble averaging. Thus we obtain the result that

$$\langle \mathcal{P}_{LT\,W/O} \rangle = \tfrac{1}{2}\lambda^{-2} R^{-2} A^{*} \iint d\vec{r}\; d\vec{r}\,'\; W(\vec{r})\; W(\vec{r}\,')$$
$$\times \langle \exp[\ell(\vec{r},0) + \ell(\vec{r}\,',0)] \rangle \qquad . \qquad (2.76)$$

It is convenient at this point to introduce the variable

$$\vec{\rho} = \vec{r}\,' - \vec{r} \qquad , \qquad (2.77)$$

which on making a change of variable of integration in Eq. (76) allows us to write

$$\langle \mathcal{P}_{LT\,W/O} \rangle = \tfrac{1}{2}\lambda^{-2} R^{-2} A^{*} A \iint d\vec{r}\; d\vec{\rho}\; W(\vec{r})\, W(\vec{r}+\vec{\rho})$$
$$\times \langle \exp[\ell(\vec{r},0) + \ell(\vec{r}+\vec{\rho},0)] \rangle \qquad . \qquad (2.78)$$

By using Eq. (54) we can rewrite this result as

$$\langle \mathcal{P}_{LT\,W/O} \rangle = \tfrac{1}{2}\lambda^{-2} R^{-2} A^{*} A \iint d\vec{r}\; d\vec{\rho}\; W(\vec{r})\, W(\vec{r}+\vec{\rho})$$
$$\times \exp[-C_{\ell}(0,0) + C_{\ell}(\vec{\rho},0)] \qquad . \qquad (2.79)$$

As noted in the discussion following Eq. (58), the log-amplitude covariance function, $C_{\ell}(\vec{\rho},0)$, is in general quite short range — much shorter range than most laser transmitter diameters of interest. Thus over most of the range of the $\vec{\rho}$ - integration the $C_{\ell}(\vec{\rho},0)$ term can be set equal to zero. This leads to the approximate result that

$$\langle \theta_{LTW/O} \rangle \approx \tfrac{1}{2}\lambda^{-2} R^{-2} A^{*} A \exp[-C_{\ell}(0,0)] \iint d\vec{r}\, d\vec{\rho}\; W(\vec{r})\, W(\vec{r}+\vec{\rho}). \tag{2.80}$$

Changing the variable of integration from $\vec{\rho}$ back to $\vec{r}\,'$, and then noting that the double integral then separates into the product of two identical integrals, each equal to $\frac{1}{4}\pi D^2$, we get

$$\langle \theta_{LTW/O} \rangle \approx \tfrac{1}{2}\lambda^{-2} R^{-2} A^{*} A \exp[-C_{\ell}(0,0)]\,(\tfrac{1}{4}\pi D^2)^2 \; . \tag{2.81}$$

When we substitute this result together with Eq. (74) into Eq. (69), and take note of Eq. (32), we get the result that

$$\begin{aligned} \langle G_{LTW/O} \rangle &\approx \tfrac{1}{4}\pi\,(D/\lambda)^2 \exp[-C_{\ell}(0,0)] \\ &\approx G_{DL} \exp[-C_{\ell}(0,0)] \qquad , \end{aligned} \tag{2.82}$$

as the ensemble average adaptive optics laser transmitter antenna gain when the adaptive optics does not provide a random apodization correction, and there is no angular separation between the directions to the target aim-point and to the reference beacon.

It is obvious from consideration of Eq. (82) that the failure to provide for the random apodization correction reduces the adaptive optics laser transmitter antenna gain by a factor of $\exp[-C_{\ell}(0,0)]$. If the log-amplitude variance, $C_{\ell}(0,0)$, is small enough this can be ignored, but for nontrivial values of the log-amplitude variance it would be desirable to have an adaptive optics that could make random apodization corrections — if possible.

Before we leave this subsection to take up the case of a laser transmitter with adaptive optics capable of producing random apodization correction, we wish to write down the relevant result when we do not make the short range of the log-amplitude covariance function approximation that we used in going from Eq. (79) to Eq. (80). The $\vec{r}$-integration in Eq. (79) can be carried out (inside the $\vec{\rho}$-integration) to give a factor of $\frac{1}{4}\pi D^2 K(\rho)$, as noted in the discussion just preceeding Eq. (21). Accordingly Eq. (79) can be written as

$$\langle \mathcal{G}_{LTW/O} \rangle = \tfrac{1}{2} \lambda^{-2} R^{-2} A^{*} A \left(\tfrac{1}{4}\pi D^2\right) \int d\vec{\rho}\ K(\rho) \times \exp\left[-C_\ell(0,0) + C_\ell(\vec{\rho},0)\right] \quad . \tag{2.83}$$

Substituting this together with Eq. (74) into Eq. (69) we get the more exact result that

$$\langle G_{LTW/O} \rangle = \lambda^{-2} \int d\vec{\rho}\ K(\rho) \exp\left[-C_\ell(0,0) + C_\ell(\vec{\rho},0)\right] . \tag{2.84}$$

Taking note of the fact that

$$\int d\vec{\rho}\ K(\rho) = \tfrac{1}{4}\pi D^2 \quad , \tag{2.85}$$

and using Eq. (32), we can rewrite Eq. (84) in the very convenient form

$$\langle G_{LTW/O} \rangle = \tfrac{1}{4}\pi (D/\lambda)^2 \exp\left[-C_\ell(0,0)\right] \frac{\int d\vec{\rho} K(\rho) \exp\left[C_\ell(\vec{\rho},0)\right]}{\int d\vec{\rho}\ K(\rho)}$$

$$= G_{0L} \exp\left[-C_\ell(0,0)\right] \left\{ \frac{\int d\vec{\rho} K(\rho) \exp\left[C_\ell(\vec{\rho},0)\right]}{\int d\vec{\rho}\ K(\rho)} \right\} . \tag{2.86}$$

To the extent that the log-amplitude covariance is very short range compared to the laser transmitter aperture diameter, D, the quantity in the curly brackets will be very nearly equal to unity — thus justifying the approximate result of Eq. (82).

With this result in hand we are now ready to take up the case of the adaptive optics laser transmitter with adaptive optics that can provide random apodization correction. This is treated in the next subsection.

1.2.2.4 Ideal Laser Transmitter Adaptive Optics With Random Apodization Correction

For the case of an adaptive optics laser transmitter with the ability to correct for random apodization Eq.'s (65), (66), and (67) remain applicable. We rewrite these with the subscript LTW as

$$\langle G_{LTW} \rangle = (\langle \mathcal{P}_{LTW} \rangle / \langle P_{LTW} \rangle) R^2 \quad , \tag{2.87}$$

$$\langle \mathcal{P}_{LTW} \rangle = \tfrac{1}{2} \langle u^*_{LTW} u_{LTW} \rangle \quad , \tag{2.88}$$

and

$$u_{LTW} = -\frac{i}{\lambda R} \int d\vec{r}\; U_{LTW}(\vec{r}) \exp\{i[k L(\vec{r}, \vec{\theta}_T) + \psi(\vec{r}, \vec{\theta}_T)]\} \; , \tag{2.89}$$

corresponding to Eq.'s (69), (70), and (71). We also note that corresponding to Eq. (73) we can write

$$\langle P_{LTW} \rangle = \int d\vec{r}\; \tfrac{1}{2} \langle U^*_{LTW}(\vec{r})\, U_{LTW}(\vec{r}) \rangle \quad . \tag{2.90}$$

The careful reader will have noted that we seem to have skipped over the task of writing down the expression corresponding to Eq. (68). It would be this expression that would indicate the nature of the random apodization correction. As we shall see this matter requires some careful consideration and the appropriate result is not at first apparent.

Off hand it would seem that the appropriate correction would be just as in the imaging case, as presented in Eq. (61) — namely the random apodization correction should cause the the pupil function to vary inversely in power density to the power density variation of the beacon signal { In Eq. (61) the factor of exp $[-\ell(\vec{r}, \vec{\theta}_B)]$ is the inverse of the beacon signal's power density, which is proportional to exp $[\ell(\vec{r}, \vec{\theta}_B)]$}. If the pupil function, $U_{LTW}(\vec{r})$ were formed in this way then the power density variation would exactly cancel the $\psi(\vec{r}, \vec{\theta}_T)$ induced variation in Eq. (89) and we would seem to be forming a diffraction limited power density, u_{LTW}, at the aim-point. However, this is not enough to insure diffraction limited laser transmitter antenna gain performance since these variations in the pupil function's power density can effect the total laser power transmitted, and antenna gain represents a ratio of power density at the target aim-point to the total laser power transmitted. We therefore must choose the nature of the random apodization correction with the average antenna gain, $\langle G_{LTW} \rangle$, in mind rather than with the target aim-point power density, $\langle \mathcal{P}_{LTW} \rangle$, in mind.

It is not clear exactly how the problem of defining the optimum random apodization correction should be attempted. It appears to be a fairly complex problem in the calculus of variations. Accordingly we shall attempt the simplier problem of finding the value of β in the random apodization correction pupil function expression

$$U_{LTW}(\vec{r}) = A W(\vec{r}) \exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}_T] - \phi(\vec{r},\vec{\theta}_B) + \beta\,\ell(\vec{r},\vec{\theta}_B)\}, \quad (2.91)$$

To do this we need to apply Eq. (91) to obtain an expression for the ensemble average laser power transmitted as a function of β, and to obtain an expression for the ensemble average laser power density at the target aim-point, also as a function of β. From these results an expression for the ensemble average antenna gain as a function of β will follow immediately and choice of an optimum value for β will be straight forward.

When we substitute Eq. (91) twice into Eq. (90) and carry out the appropriate simplifications we get the result that the average laser power transmitted is

$$\langle P_{LTW} \rangle = \tfrac{1}{2} A^* A \int d\vec{r}\; W(\vec{r}) \langle \exp[2\beta\ell(\vec{r},\vec{\theta}_B)] \rangle \quad . \qquad (2.92)$$

Making use of Eq.'s (35), (37), and (47) we can write

$$\begin{aligned} \langle \exp[2\beta\,\ell(\vec{r},\vec{\theta}_B)] \rangle &= \exp(2\beta\,\bar{\ell})\exp[2\beta^2 \langle (\ell - \bar{\ell})^2 \rangle] \\ &= \exp[(2\beta^2 - 2\beta)\langle(\ell-\bar{\ell})^2\rangle] \\ &= \exp[(2\beta^2 - 2\beta)C_\ell(0,0)] \quad . \qquad (2.93) \end{aligned}$$

By virtue of the stationarity of the statistics of the log-amplitude this is obviously not a function of aperture position, $\vec{r}$. Accordingly, when Eq. (93) is substituted into Eq. (92) the $\vec{r}$-integration reduces to the evaluation of the area of a circle of diameter D, so that we get

$$\langle P_{LTW} \rangle = \tfrac{1}{2} A^* A (\tfrac{1}{4}\pi D^2) \exp[(2\beta^2 - 2\beta)C_\ell(0,0)] \quad , \qquad (2.94)$$

for the ensemble average transmitted laser power.

To evaluate the laser power density at the target aim-point, $\langle P_{LTW} \rangle$, we first substitute Eq. (91) into Eq. (89). Appropriately simplifying we get the expression

$$u_{LTW} = -\frac{i}{\lambda R} \exp\left[ikR(1+\tfrac{1}{2}\theta_T{}^2)\right] A \int d\vec{r}\ W(\vec{r})$$

$$\times \exp\{i[\phi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)] + \ell(\vec{r},\vec{\theta}_T) + \beta\,\ell(\vec{r},\vec{\theta}_B)\}. \quad (2.95)$$

At this point we recall that we are considering the idealized case where there is no angular separation between the directions to the target aim-point and to the reference beacon, both being on the z-axis, so that $\vec{\theta}_T = \vec{\theta}_B = 0$. This allows us to reduce Eq. (95) to the simplified form

$$u_{LTW} = -\frac{i}{\lambda R} \exp(ikR)\ A \int d\vec{r}\ W(\vec{r}) \exp[(1+\beta)\,\ell(\vec{r},0)]. \quad (2.96)$$

When we substitute this result into Eq. (88) and suitably simplify, we get for the ensemble average laser power density at the target aim-point

$$\langle \mathcal{P}_{LTW} \rangle = \tfrac{1}{2} A^* A\, \lambda^{-2} R^{-2} \iint d\vec{r}\ d\vec{r}'\ W(\vec{r})\ W(\vec{r}')$$

$$\times \langle \exp\{(1+\beta)[\ell(\vec{r},0) + \ell(\vec{r}',0)]\}\rangle. \quad (2.97)$$

Once again making use of Eq.'s (35), (37), and (47), we can now write

$$\langle \exp\{(1+\beta)[\ell(\vec{r},0)+\ell(\vec{r}',0)]\}\rangle = \exp[2(1+\beta)\,\bar{\ell}\,]$$

$$\times \exp\{(1+\beta)^2[C_\ell(0,0) + C_\ell(\vec{r}'-\vec{r},0)]\}$$

$$= \exp[(\beta^2-1)C_\ell(0,0) + (1+\beta)^2 C_\ell(\vec{r}'-\vec{r},0)]. \quad (2.98)$$

When we substitute this result into Eq. (97) and replace the variable of integration $\vec{r}'$ with $\vec{\rho}$, [with $\vec{\rho}$ as defined by Eq. (72)], we get

$$\langle \mathcal{P}_{LTW} \rangle = \tfrac{1}{2} A^* A\, \lambda^{-2} R^{-2} \iint d\vec{r}\ d\vec{\rho}\ W(\vec{r})\ W(\vec{r}+\vec{\rho})$$

$$\times \exp[(\beta^2-1)C_\ell(0,0) + (1+\beta)^2 C_\ell(\vec{\rho},0)]. \quad (2.99)$$

Introducing the approximation that since the log-amplitude covariance function, $C_\ell(\vec{\rho},0)$, is a short range function in terms of the aperture diameters of interest to us, we can consider $C_\ell(\vec{\rho},0)$ to have zero value over most of the range of integration, we can write

$$\langle \mathcal{P}_{LTW} \rangle \approx \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \exp[(\beta^2 - 1)\, C_\ell(0,0)] \iint d\vec{r}\, d\vec{\rho}\, W(\vec{r})\, W(\vec{r}+\vec{\rho})$$

$$\approx \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \exp[(\beta^2 - 1)\, C_\ell(0,0)] \iint d\vec{r}\, d\vec{r}'\, W(\vec{r})\, W(\vec{r}')$$

$$\approx \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \exp[(\beta^2 - 1)\, C_\ell(0,0)]\, (\tfrac{1}{4}\pi D^2)^2 \; . \tag{2.100}$$

Obviously, either $\beta = +1$ or $\beta = -1$ will result in an ensemble average laser power density of the target aim-point which is independent of the turbulence induced log-amplitude variance. But, the ensemble average laser power transmitted does depend on β, as we can see from Eq. (94). The ensemble average antenna gain, which is what we want to optimize by our choice of β, is obtained by substituting Eq.'s (94) and (100) into Eq. (87). This yields the result that

$$\langle G_{LTW} \rangle \approx \tfrac{1}{4}\pi (D/\lambda)^2 \exp[(-\beta^2 + 2\beta - 1)\, C_\ell(0,0)]$$

$$\approx G_{DL} \exp[-(\beta - 1)^2\, C_\ell(0,0)] \; . \tag{2.101}$$

It is obvious from this expression that the ensemble average antenna gain, $\langle G_{LTW} \rangle$, will be maximized [and that this maximum will be equal to the diffraction limited value of antenna gain, G_{DL}, as defined by Eq. (32)] when β equals unity.

If we go back to Eq. (99) and do not make the approximation that the log-amplitude covariance is a short range function relation to the aperture diameters of interest then using Eq. (21) to allow the $\vec{r}$-integration to be performed, we get

$$\langle \mathcal{P}_{LTW} \rangle = \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} (\tfrac{1}{4}\pi D^2) \exp[(\beta^2 - 1)\, C_\ell(0,0)]$$

$$\times \int d\vec{\rho}\, K(\rho) \exp[(1+\beta)^2\, C_\ell(\vec{\rho}, 0)] \; . \tag{2.102}$$

Substituting this together with Eq. (94) into Eq. (87) we get as a more exact expression for the ensemble average antenna gain

$$\langle G_{LTW} \rangle = \lambda^{-2} \exp[-(\beta - 1)^2\, C_\ell(0,0)]$$

$$\times \int d\rho K(\rho) \exp[(1+\beta)^2\, C_\ell(\vec{\rho}, 0)] \; . \tag{2.103}$$

Making use of Eq.'s (32) and (85) this result can be cast in the convenient form

$$\langle G_{LTW} \rangle = G_{DL} \exp[-(\beta-1)^2 C_\ell(0,0)] \left\{ \frac{\int d\vec{\rho}\, K(\rho) \exp[(1+\beta)^2 C_\ell(\rho,0)]}{\int d\vec{\rho}\, K(\rho)} \right\} . \tag{2.104}$$

The surprising thing to be noted from this expression for the ensemble average antenna gain is that while for large enough aperture diameters the quantity in the curly brackets is very nearly unity so that the optimum choice of value of β is unity and the maximum possible value of antenna gain is the diffraction limited value, for small enough values of the aperture diameter the situation is quite different. For very small values of aperture diameter the ensemble average antenna gain becomes very nearly equal to $G_{DL} \exp[4\beta C_\ell(0,0)]$. In this case very large values of β appear desirable and the ensemble average antenna gain can be greater than the diffraction limited value. However, this is a matter of having the laser transmitter emit the largest amount of power at those instances when the power density of the beacon signal indicates that the "connectivity" between the laser transmitter aperture and the target aim-point provided by the turbulence is greatest. Though it seems interesting to note here this somewhat surprising possibility, we believe that for a number of practical reasons the possibility of exploiting this feature is relatively uninteresting.

Before leaving the subject area of laser transmitter adaptive optics with random apodization correction capability it is perhaps worth remarking on the implications of having $\rho = 1$ in Eq. (91). This means that we have

$$U_{LTW}(\vec{r}) = A W(\vec{r}) \exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}_T] - i\phi(\vec{r},\vec{\theta}_B) + \ell(\vec{r},\vec{\theta}_B)\} . \tag{2.105}$$

Making use of Eq. (34) we can rewrite this as

$$U_{LTW}(\vec{r}) = A W(\vec{r}) \exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}_T] - i\overset{*}{\psi}(\vec{r},\vec{\theta}_B)\} . \tag{2.106}$$

Comparing this with Eq. (62) for the imaging system with random apodization compensation, we see that while the adaptive optics of the imaging system

results in subtraction of the complex phase of the beacon, for the laser transmitter it is the conjugate of the complex phase of the beacon that is subtracted.

With all of these results in hand for imaging system and laser transmitter performance under the ideal conditions of no angular separation between the directions from the aperture to the target and to the beacon, with both target and beacon lying on the z-axis, we are now ready to turn our attention to the cases of basic interest, when the angular separation is non-zero and there is a possibility of anisoplanatism effects. However, before we can evaluate system performance in such a situation, we must first develop expressions for the joint statistics for propagation in two distinct directions. This is taken up in the next section.

1.3 Propagation Theory With Angular Separation of Paths

The analytic foundation for the propagation theory that we shall develop in this section will be found in an earlier paper of ours[6] titled "Spectral and Angular Covariance of Scintillation for Propagation in a Randomly Inhomogeneous Medium". In the following we shall refer to this paper as SAC. The portion of SAC concerned with the angular covariance is directly relevant to our requirements here, though it needs some extensions and modifications to provide the propagation theory basis for the work undertaken here. (There is, in addition a requirement for one non-trivial correction.) While the results of SAC are given for the log-amplitude and for the phase covariance, no results are given for the cross-statistics between phase and log-amplitude. We shall need this latter result to establish the isotropy of the cross-statistics — a fact which we shall use later and which is not apparent from any generalized reasoning. In addition, while the results of SAC are developed for infinite plane wave propagation, to allow us to apply our work to the case in which the adaptive optics system and the target/beacon are within the atmosphere, we shall need corresponding results for spherical wave propagation theory. Our approach to developing the spherical wave results will be heuristic arguments extending the infinite plane wave results. It will be the detailed development of the infinite plane wave results that we shall concentrate on, and only at the very end of this section will we introduce the heuristically justified modifications that will make these results applicable to the spherical wave case.

In what follows we shall not repeat the portion of SAC concerned with the equations for propagation through a random medium and the development of the random wavefunction. We shall simply quote these results and then apply them to the development of the statistics. We shall, however, go over most of the notation as we wish to make some (simplifying) changes in the notation. We start, in the next subsection with a presentation of the notation.

1.3.1 Notation and Random Propagation Results from SAC

The basic random function of interest to us is the previously introduced complex phase, $\psi(\vec{r}, \vec{\theta})$, with its real and imaginary parts, $\phi(\vec{r}, \vec{\theta})$, the real phase, and $\ell(\vec{r}, \vec{\theta})$, the log-amplitude variance, which by copying Eq. (2.34) we write here as

$$\psi(\vec{r}, \vec{\theta}) = \phi(\vec{r}, \vec{\theta}) - i\ell(\vec{r}, \vec{\theta}) \quad . \tag{3.1}$$

Here $\vec{r}$ denotes a position on the aperture plane at which the complex phase is measured, and $\vec{\theta}$ indicates the direction from the center of the aperture to a monochromatic point source of wavelength λ at a z-axis distance equal to R. The angle $\vec{\theta}$ is a measure of the deviation of the direction from the z-axis, which axis is perpendicular to the aperture plane. (For a point source on the z-axis the angle $\vec{\theta}$ is equal to zero.) In addition to the complex phase, ψ, the other basic random function which we have to consider is $n(\vec{r}, z)$, the random deviation of the refractive-index from its nominal mean value of unity. (It is this random function that we are referring to when we speak of turbulence.) The refractive-index denoted by $n(\vec{r}, z)$ is measured at a distance z from the aperture plane along the z-axis, and at a position $\vec{r}$ displaced from the z-axis.

In Eq. (2.47) we introduced the covariance of the log-amplitude, C_ℓ. We repeat that definition here in a slightly modified form along with definitions of all the other statistical function we shall be concerned with. These are as follows; the log-amplitude mean value

$$\bar{\ell} = \langle \ell(\vec{r}, \vec{\theta}) \rangle \quad , \tag{3.2}$$

the real phase mean value

$$\bar{\phi} = \langle \phi(\vec{r}, \vec{\theta}) \rangle \quad , \tag{3.3}$$

the log-amplitude convariance

$$C_\ell(\vec{\rho}, \vec{\vartheta}) = \langle [\ell(\vec{r} + \vec{\rho}, \tfrac{1}{2}\vec{\vartheta}) - \bar{\ell}][\ell(\vec{r}, -\tfrac{1}{2}\vec{\vartheta}) - \bar{\ell}] \rangle, \tag{3.4}$$

the phase covariance

$$C_\phi(\vec{\rho}, \vec{\vartheta}) = \langle [\phi(\vec{r}+\vec{\rho}, \tfrac{1}{2}\vec{\vartheta}) - \bar{\phi}][\phi(\vec{r}, -\tfrac{1}{2}\vec{\vartheta}) - \bar{\phi}] \rangle , \qquad (3.5)$$

the log-amplitude: phase cross-covariances

$$C_{\ell}(\vec{\rho}, \vec{\vartheta}) = \langle [\ell(\vec{r}+\vec{\rho}, \tfrac{1}{2}\vec{\vartheta}) - \bar{\ell}][\phi(\vec{r}, -\tfrac{1}{2}\vec{\vartheta}) - \bar{\phi}] \rangle , \qquad (3.6)$$

and the refractive-index covariance

$$C_N(\vec{\rho}, \xi) = \langle n(\vec{r}+\vec{\rho}, z+\xi)\, n(\vec{r}, z) \rangle \qquad . \qquad (3.7)$$

We shall have occasion to work with the Fourier transforms of the above four covariance functions. For the log-amplitude covariance, phase covariance, and log-amplitude:phase cross-covariance these are

$$F_{\ell}(\vec{\sigma}, \vec{\vartheta}) = \int d\vec{\rho}\, C_{\ell}(\vec{\rho}, \vec{\vartheta}) \exp(-i\vec{\sigma}\cdot\vec{\rho}) \qquad , \qquad (3.8)$$

$$F_\phi(\vec{\sigma}, \vec{\vartheta}) = \int d\vec{\rho}\, C_\phi(\vec{\rho}, \vec{\vartheta}) \exp(-i\vec{\sigma}\cdot\vec{\rho}) \qquad , \qquad (3.9)$$

and

$$F_{\ell\phi}(\vec{\sigma}, \vec{\vartheta}) = \int d\vec{\rho}\, C_{\ell\phi}(\vec{\rho}, \vec{\vartheta}) \exp(-i\vec{\sigma}\cdot\vec{\rho}) \qquad , \qquad (3.10)$$

respectively. For the refractive-index covariance we shall consider both a two-dimensional Fourier transform, namely

$$F_N(\vec{\sigma}, \xi) = \int d\vec{\rho}\, C_N(\vec{\rho}, \xi) \exp(-i\vec{\sigma}\cdot\vec{\rho}) \qquad , \qquad (3.11)$$

and a three-dimensional transform

$$\mathfrak{F}_N(\vec{\sigma}, s) = \iint d\vec{\rho}\, d\xi\, C_N(\vec{\rho}, \xi) \exp[-i(\vec{\sigma}\cdot\vec{\rho} + s\xi)] \ . \qquad (3.12)$$

It is immediately obvious from consideration of Eq.'s (11) and (12) that

$$\mathfrak{F}_N(\vec{\sigma}, s) = \int d\xi\, F_N(\vec{\sigma}, \xi) \exp(-is\xi) \qquad . \qquad (3.13)$$

As noted in SAC experimental[7] and theoretical evidence[8, 9] based on the Kolmogoroff's theory of turbulence in the inertial subrange and Tatarski's arguments of a conserved passive additive nature for the optical aspects of turbulence lead to the conclusion that

$$\mathfrak{F}_N(\vec{\sigma}, s) = 8.16\, C_N^2 (\sigma^2 + s^2)^{-11/6} \qquad , \qquad (3.14)$$

where C_N^2 is a scalar quantity called the refractive-index structure constant which serves as a measure of the local optical strength of turbulence. (The dimensionality of C_N^2 is length to the minus two-thirds power.)

In addition to these Fourier transforms of statistical functions SAC introduces, we shall have to make use of Fourier transforms of certain random functions. These Fourier transforms are as follows: for the complex phase we have

$$\Psi(\vec{\sigma},\vec{\theta}) = \int d\vec{r}\ \psi(\vec{r},\vec{\theta})\ \exp(-i\vec{\sigma}\cdot\vec{r}) \quad , \qquad (3.15)$$

for the log-amplitude we have

$$L(\vec{\sigma},\vec{\theta}) = \int d\vec{r}\ \ell(\vec{r},\vec{\theta})\ \exp(-i\vec{\sigma}\cdot\vec{r}) \quad , \qquad (3.16)$$

for the real phase we have

$$\Phi(\vec{\sigma},\vec{\theta}) = \int d\vec{r}\ \phi(\vec{r},\vec{\theta})\ \exp(-i\vec{\sigma}\cdot\vec{r}) \quad , \qquad (3.17)$$

and for the refractive-index we have

$$N(\vec{\sigma}, z) = \int d\vec{r}\ n(\vec{r}, z)\ \exp(-i\vec{\sigma}\cdot\vec{r}) \quad . \qquad (3.18)$$

It follows directly from Eq.'s (1), (15), that

$$L(\vec{\sigma},\vec{\theta}) = \tfrac{1}{2}i[\Psi(\vec{\sigma},\vec{\theta}) - \Psi^*(-\vec{\sigma},\vec{\theta})] \quad , \qquad (3.19)$$

and from Eq.'s (1), (15), and (17), that

$$\Phi(\vec{\sigma},\vec{\theta}) = \tfrac{1}{2}[\Psi(\vec{\sigma},\vec{\theta}) + \Psi^*(-\vec{\sigma},\vec{\theta})] \quad . \qquad (3.20)$$

The solution of the propagation equation carried out in SAC leads to the result that*

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* Eq. (21) above is in some significant ways different from Eq. (20) in SAC from which it is taken. Some of these differences are due to the fact that Ψ as given by Eq. (1) here is equal to Ψ as defined in SAC times i. A second cause of difference is that while in SAC the integration runs from the source to the plane where Ψ is measured (with the variable of integration being equal to zero at the source and equal to R at the measurement plane), here the integration runs from zero at the (aperture) plane where Ψ is measured to $z = R$ at the source.

$$\Psi(\vec{\sigma},\vec{\theta}) = 2ik\int_0^R dz\; N(\vec{\sigma},z)\;\exp(-ikz)\sin[\gamma(\vec{\sigma},\vec{\theta})z], \quad (3.21)$$

where

$$\gamma(\vec{\sigma},\vec{\theta}) = k - \tfrac{1}{2}\frac{\sigma^2}{k} - \vec{\sigma}\cdot\vec{\theta} - \tfrac{1}{2}k\theta^2 \quad . \quad (3.22)$$

Before leaving this subsection it will be convenient to rewrite Eq. (21) as

$$\Psi(\vec{\sigma},\vec{\theta}) = -k\int_0^R dz\; N(\vec{\sigma},z)\Big(\exp\{i[\gamma(\vec{\sigma},\vec{\theta})-k]z\} - \exp\{-i[\gamma(\vec{\sigma},\vec{\theta})+k]z\}\Big). \quad (3.23)$$

and from this to write

$$\Psi^*(-\vec{\sigma},\vec{\theta}) = -k\int_0^R dz\; N^*(-\vec{\sigma},z)\Big(\exp\{-i[\gamma(-\vec{\sigma},\vec{\theta})-k]z\} - \exp\{i[\gamma(-\vec{\sigma},\vec{\theta})+k]z\}\Big). \quad (3.24)$$

By virtue of the fact that the refractive-index variation is real (so that $n^* = n$) it follows from Eq. (18) that

$$N^*(-\vec{\sigma},z) = N(\vec{\sigma},z) \quad , \quad (3.25)$$

so that we can rewrite Eq. (24) as

$$\Psi^*(-\vec{\sigma},\vec{\theta}) = -k\int_0^R dz\; N(\vec{\sigma},z)\Big(\exp\{-i[\gamma(-\vec{\sigma},\vec{\theta})-k]z\} - \exp\{i[\gamma(-\vec{\sigma},\vec{\theta})+k]z\}\Big). \quad (3.26)$$

When Eq.'s (23) and (26) are substituted into Eq.'s (19) and (20) and the obvious simplifications are made, we get the results that

$$\begin{aligned} L(\vec{\sigma},\vec{\theta}) = -\tfrac{1}{2}ik\int_0^R dz\; N(\vec{\sigma},z)\Big(&\exp\{i[\gamma(\vec{\sigma},\vec{\theta})-k]z\} \\ &-\exp\{-i[\gamma(\vec{\sigma},\vec{\theta})+k]z\} - \exp\{-i[\gamma(-\vec{\sigma},\vec{\theta})-k]z\} \\ &+\exp\{i[\gamma(-\vec{\sigma},\vec{\theta})+k]z\}\Big) \quad , \end{aligned} \quad (3.27)$$

and

$$\Phi(\vec{\sigma},\vec{\theta}) = -\tfrac{1}{2} i k \int_0^R dz\, N(\vec{\sigma}, z) \Big(\exp\{ i [\gamma(\vec{\sigma},\vec{\theta}) - k] z \}$$

$$- \exp\{ -i [\gamma(\vec{\sigma},\vec{\theta}) + k] z \} + \exp\{ -i [\gamma(-\vec{\sigma},\vec{\theta}) - k] z \}$$

$$- \exp\{ i [\gamma(-\vec{\sigma},\vec{\theta}) + k] z \} \Big) \quad . \qquad (3.28)$$

At this point we note that of the four exponentials in each of the above two integrands, as we can see from consideration of Eq. (22), and of the facts that k is much greater than σ and that the magnitude of $\vec{\theta}$ is very small, two of the exponentials oscillate very rapidly (at a "rate" of about $2k$), while the other two oscillate much more slowly. Such rapid oscillations can not contribute to the value of the integrals and accordingly we can make the approximations that

$$L(\vec{\sigma},\vec{\theta}) = -\tfrac{1}{2} i k \int_0^R dz\, N(\vec{\sigma}, z) \Big(\exp\{ i [\gamma(\vec{\sigma},\vec{\theta},) - k] z \}$$

$$- \exp\{ -i [\gamma(-\vec{\sigma},\vec{\theta}) - k] z \} \Big) \quad , \qquad (3.29)$$

and

$$\Phi(\vec{\sigma},\vec{\theta}) = -\tfrac{1}{2} k \int_0^R dz\, N(\vec{\sigma}, z) \Big(\exp\{ i [\gamma(\vec{\sigma},\vec{\theta}) - k] z \}$$

$$+ \exp\{ -i [\gamma(-\vec{\sigma},\vec{\theta}) - k] z \} \Big) \quad . \qquad (3.30)$$

Making use of Eq. (22) we can recast these results in the form

$$L(\vec{\sigma},\vec{\theta}) = -\tfrac{1}{2} i k \int_0^R dz\, N(\vec{\sigma}, z) \Big(\exp\{ [-i (\tfrac{1}{2} k \theta^2 + \tfrac{1}{2} \sigma^2/k) - i \vec{\sigma}\cdot\vec{\theta}] z \}$$

$$- \exp\{ [i (\tfrac{1}{2} k \theta^2 + \tfrac{1}{2} \sigma^2/k) - i \vec{\sigma}\cdot\vec{\theta}] z \} \Big) \quad , \qquad (3.31)$$

$$\Phi(\vec{\sigma},\vec{\theta}) = -\tfrac{1}{2} k \int_0^R dz\, N(\vec{\sigma}, z) \Big(\exp\{ [-i (\tfrac{1}{2} k \theta^2 + \tfrac{1}{2} \sigma^2/k) - i \vec{\sigma}\cdot\vec{\theta}] z \}$$

$$+ \exp\{ [i (\tfrac{1}{2} k \theta^2 + \tfrac{1}{2} \sigma^2/k) - i \vec{\sigma}\cdot\vec{\theta}] z \} \Big) \quad . \qquad (3.32)$$

This is as far as we can go with the analysis of the random propagation functions. To proceed beyond this point we have to start working with the statistical functions. To do this we need to make use of a special theorem developed in SAC. This is presented in the next subsection.

1.3.2 Fourier Transform Statistics

In this brief subsection we wish to prove a theorem about the ensemble average of the product of Fourier transforms of random functions. We will show that an integral over such an average gives rise to a power spectrum (or cospectrum) for the (two) random functions, with a Dirac delta function like property associated with the average.

We consider two real, stationary random functions, $f(\vec{x};\vec{\alpha})$ and $g(\vec{x};\vec{\beta})$, where $\vec{x}$ is an N-dimensional variable and $\vec{\alpha}$ and $\vec{\beta}$ are some multicomponent parameter "vectors". (The functions f and g need not be distinct, but for generality we allow them to be so.) We define the cross-covariance as

$$C_{FG}(\vec{x}'-\vec{x};\vec{\alpha},\vec{\beta}) = \langle [f(\vec{x}';\vec{\alpha}) - \bar{f}(\alpha)][g(\vec{x};\vec{\beta}) - \bar{g}(\beta)] \rangle , \qquad (3.33)$$

where

$$\bar{f}(\vec{\alpha}) = \langle f(\vec{x};\vec{\alpha}) \rangle \quad , \qquad (3.34)$$

$$\bar{g}(\vec{\beta}) = \langle g(\vec{x};\beta) \rangle \quad . \qquad (3.35)$$

In addition we write the Fourier transforms of the two random functions as

$$F(\vec{y},\vec{\alpha}) = \int d\vec{x}\,[f(\vec{x};\vec{\alpha}) - \bar{f}(\vec{\alpha})] \exp(-i\vec{x}\cdot\vec{y}) \quad , \qquad (3.36)$$

$$G(\vec{y},\vec{\beta}) = \int d\vec{x}\,[g(\vec{x};\vec{\beta}) - \bar{g}(\vec{\beta})] \exp(-i\vec{x}\cdot\vec{y}) \quad . \qquad (3.37)$$

We also note that we can write the cospectrum associated with these two random functions as

$$\Gamma_{FG}(\vec{y};\vec{\alpha},\vec{\beta}) = \int d\vec{x}\; C_{FG}(\vec{x};\vec{\alpha},\vec{\beta}) \exp(-i\vec{x}\cdot\vec{y}) \quad . \qquad (3.38)$$

We now introduce the function $h(\vec{y})$ to represent any nonrandom, reasonably well behaved function and consider the problem of evaluating the expression

$$\int d\vec{y}' \langle F^*(\vec{y}';\vec{\alpha})\, G(\vec{y};\vec{\beta}) \rangle \; h(\vec{y}') \quad .$$

To carry out the evaluation we start by making use of Eq.'s (36) and (37). This allows us to write

$$\int d\vec{y}' \langle F^*(\vec{y}';\vec{\alpha})\, G(\vec{y};\vec{\beta}) \rangle\, h(\vec{y}') = \iiint d\vec{x}\; d\vec{x}'\; d\vec{y}'$$

$$\times \langle [f(\vec{x}';\vec{\alpha}) - \bar{f}(\vec{\alpha})]\,[g(\vec{x};\vec{\beta}) - \bar{g}(\vec{\beta})] \rangle\, h(\vec{y}') \exp[-i(\vec{x}\cdot\vec{y} - \vec{x}'\cdot\vec{y}')]. \tag{3.38}$$

At this point we introduce sum and difference coordinates

$$\vec{p} = \vec{x}' - \vec{x} \quad ,$$

$$\vec{q} = \tfrac{1}{2}(\vec{x}' + \vec{x}) \quad ,$$

and make these the variables of integration instead of $\vec{x}$ and $\vec{x}'$, in Eq. (39). Doing this and making use of Eq. (33) we can now write

$$\int d\vec{y}' \langle F^*(\vec{y}';\vec{\alpha})\, G(\vec{y};\vec{\beta}) \rangle\, h(\vec{y}') = \iiint d\vec{p}\; d\vec{q}\; d\vec{y}'$$

$$\times C_{FG}(\vec{p};\vec{\alpha},\vec{\beta})\, h(\vec{y}') \exp\left[i\vec{p}\cdot\left(\frac{\vec{y}'+\vec{y}}{2}\right) + i\vec{q}\cdot(\vec{y}'-\vec{y})\right], \tag{3.40}$$

where the exponential's argument has been rearranged according to the formula

$$ab - cd = (a + c)\left(\frac{b-d}{2}\right) + \left(\frac{a-c}{2}\right)(b + d) \quad . \tag{3.41}$$

Now we take note of the fact that according to Eq. (2.11) we can write

$$\iint d\vec{q}\; d\vec{y}'\,[h(\vec{y}') \exp(i\vec{p}\cdot\vec{y}'/2)] \exp[i\vec{q}\cdot(\vec{y}'-\vec{y})]$$

$$= (2\pi)^N\, h(\vec{y}) \exp(i\vec{p}\cdot\vec{y}/2) \quad , \tag{3.42}$$

where N, we recall, is the dimensionality of our variables. When we substitute Eq. (42) into Eq. (40) and simplify as appropriate we obtain the result that

$$\int d\vec{y}' \langle F^*(\vec{y}';\vec{\alpha})\, G(\vec{y};\vec{\beta}) \rangle\, h(\vec{y}') = (2\pi)^N h(\vec{y})$$

$$\times \int d\vec{p}\;\; C_{FG}(\vec{p};\vec{\alpha},\vec{\beta}) \exp(i\vec{p}\cdot\vec{y}) . \tag{3.43}$$

Making reference to Eq. (38) and noting that since f and g are real functions then so is C_{FG}, we see that the integral on the right-hand-side of Eq. (43)

can be identified with Γ_{FG}^{*}. Thus we can write

$$\int d\vec{y}' \langle F^{*}(\vec{y}';\vec{\alpha})G(\vec{y};\vec{\beta})\rangle h(\vec{y}') = (2\pi)^{N} h(\vec{y})\, \Gamma_{FG}^{*}(\vec{y};\vec{\alpha},\vec{\beta}). \quad (3.44)$$

This is the theorem we wished to prove in this subsection.

Three alternate forms of this result are as follows:

$$\int d\vec{y}' \langle F(\vec{y}';\vec{\alpha})\, G^{*}(\vec{y};\vec{\beta})\rangle h(\vec{y}') = (2\pi)^{N} h(\vec{y})\, \Gamma_{FG}(\vec{y};\vec{\alpha},\vec{\beta}), \quad (3.45)$$

$$\int d\vec{y}' \langle F^{*}(\vec{y};\vec{\alpha})G(\vec{y}';\vec{\beta})\rangle h(\vec{y}') = (2\pi)^{N} h(\vec{y})\, \Gamma_{FG}(\vec{y};\vec{\alpha},\vec{\beta}), \quad (3.46)$$

and

$$\int d\vec{y}' \langle F(\vec{y},\vec{\alpha})\, G^{*}(\vec{y}';\vec{\beta})\rangle h(\vec{y}') = (2\pi)^{N} h(\vec{y})\, \Gamma_{FG}(\vec{y};\vec{\alpha},\vec{\beta}). \quad (3.47)$$

Eq. (46) is obtained by noting that replacing $F^{*}(\vec{y}';\vec{\alpha})\, G(\vec{y};\vec{\beta})$ by $F^{*}(\vec{y};\vec{\alpha})\, G(\vec{y}';\vec{\beta})$ would have altered the right hand side of Eq. (40) only to the extent of interchanging $\vec{y}$ and $\vec{y}'$ in the exponential, which as we can see from consideration of Eq. (2.11) would result in no change at all in Eq. (44). Eq. (45) is obtained as the complex conjugate of Eq. (44), while Eq. (47) is obtained in the same way from Eq. (46).

With these results in hand we are now ready to develop statistical propagation results from our random propagation expressions. We take this up in the next subsection.

1.3.3 Statistical Propagation Results

We can write the log-amplitude covariance function as the inverse Fourier transform of F_{ℓ} as defined by Eq. (8). We can see that this is to be written as

$$C_{\ell}(\vec{\rho},\vec{\vartheta}) = (2\pi)^{-2} \int d\vec{\sigma}\; F_{\ell}(\vec{\sigma},\vec{\vartheta}) \exp(i\vec{\sigma}\cdot\vec{\rho}) \quad . \quad (3.48)$$

Similarly we can write for the phase covariance

$$C_{\phi}(\vec{\rho},\vec{\vartheta}) = (2\pi)^{-2} \int d\vec{\sigma}\; F_{\phi}(\vec{\sigma},\vec{\vartheta}) \exp(i\vec{\sigma}\cdot\vec{\rho}) \quad , \quad (3.49($$

and for log-amplitude: phase cross-covariance

$$C_{\ell\phi}(\vec{\rho},\vec{\vartheta}) = (2\pi)^{-2} \int d\vec{\sigma}\; F_{\ell\phi}(\vec{\sigma},\vec{\vartheta}) \exp(i\vec{\sigma}\cdot\vec{\rho}) \quad . \quad (3.50)$$

Obviously what we need to be able to evaluate these three (cross-covariance functions is expressions for the corresponding Fourier transforms, and these we can develop by a judicious combination of the results of the two preceding subsections.

Making use of Eq.'s (45) and (27) we can write

$$(2\pi)^2\, F_\ell(\vec{\sigma},\vec{\vartheta}) = \int d\vec{\sigma}'\, \langle L(\vec{\sigma}',\tfrac{1}{2}\vec{\vartheta})\, L^*(\vec{\sigma},-\tfrac{1}{2}\vec{\vartheta})\rangle . \qquad (3.51)$$

Now making use of Eq. (31) we can recast this in the form

$$F_\ell(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{4}k/\pi)^2 \int_0^R\!\!\int_0^R dz'\,dz \int d\vec{\sigma}'\, \langle N(\vec{\sigma}',z')\, N^*(\vec{\sigma},z)\rangle$$
$$\times(\exp\{[-i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma'^2/k)-\tfrac{1}{2}i\vec{\sigma}'\cdot\vec{\vartheta}]z'\}$$
$$-\exp\{[i(\tfrac{1}{8}k\,\vartheta^2+\tfrac{1}{2}\sigma'^2/k)-\tfrac{1}{2}i\vec{\sigma}'\cdot\vec{\vartheta}]z'\})$$
$$\times(\exp\{[i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma^2/k)-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}]z\}$$
$$-\exp\{[-i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma^2/k)-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}]z\}). \qquad (3.52)$$

Proceeding in exactly the same way, we also write

$$(2\pi)^2\, F_\phi(\vec{\sigma},\vec{\vartheta}) = \int d\vec{\sigma}'\, \langle \Phi(\vec{\sigma}',\tfrac{1}{2}\vec{\vartheta})\, \Phi^*(\vec{\sigma},-\tfrac{1}{2}\vec{\vartheta})\rangle, \qquad (3.53)$$

and

$$(2\pi)^2\, F_{\ell\phi}(\vec{\sigma},\vec{\vartheta}) = \int d\vec{\sigma}'\, \langle L(\vec{\sigma}',\tfrac{1}{2}\vec{\vartheta})\, \Phi^*(\vec{\sigma},-\tfrac{1}{2}\vec{\vartheta})\rangle . \qquad (3.54)$$

Now making use of Eq. (32) as well as Eq. (31) we can recast these as

$$F_\phi(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{4}k/\pi)^2 \int_0^R\!\!\int_0^R dz'\,dz \int d\vec{\sigma}'\, \langle N(\vec{\sigma}',z')\, N^*(\vec{\sigma},z)\rangle$$
$$\times(\exp\{[-i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma'^2/k)-\tfrac{1}{2}i\vec{\sigma}'\cdot\vec{\vartheta}]z'\}$$
$$+\exp\{[i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma'^2/k)-\tfrac{1}{2}i\vec{\sigma}'\cdot\vec{\vartheta}]z'\})$$
$$\times(\exp\{[i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma^2/k)-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}]z\}$$
$$+\exp\{[-i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma^2/k)-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}]z\}), \qquad (3.55)$$

and

$$F_{\ell\phi}(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{4}k/\pi)^2 \int_0^R\!\!\int_0^R dz'\,dz \int d\vec{\sigma}' \langle N(\vec{\sigma}',z')N^*(\vec{\sigma},z)\rangle$$

$$\times\Big(\exp\{[-i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma'^2/k)-\tfrac{1}{2}i\vec{\sigma}'\cdot\vec{\vartheta}]z'\}$$

$$-\exp\{[i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma'^2/k)-\tfrac{1}{2}i\vec{\sigma}'\cdot\vec{\vartheta}]z'\}\Big)$$

$$\times\Big(\exp\{[i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}\sigma^2/k)-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}]z\}$$

$$+\exp\{[-i(\tfrac{1}{8}k\vartheta^2+\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}]z\}\Big) \quad . \qquad (3.56)$$

From consideration of Eq.'s (11) and (45) it is easy to see that

$$\int d\vec{\sigma}'\langle N(\vec{\sigma}',z')N^*(\vec{\sigma},z)\rangle E(\vec{\sigma}') = (2\pi)^2 F_N(\vec{\sigma},z'-z)E(\vec{\sigma}), \qquad (3.57)$$

where $E(\vec{\sigma})$ is any well behaved function. Making use of this result we can, with trivial effort carry out the $\vec{\sigma}'$- integrations in Eq.'s (52), (55), and (56). Thus we can write

$$F_\ell(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R\!\!\int_0^R dz'\,dz\,F_N(\vec{\sigma},z'-z)\exp[-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}(z'+z)]$$

$$\times\{\exp[-i\mu(z'-z)]-\exp[-i\mu(z'+z)]$$

$$-\exp[i\mu(z'+z)]+\exp[i\mu(z'-z)]\}\ , \qquad (3.58)$$

$$F_\phi(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R\!\!\int_0^R dz'\,dz\,F_N(\vec{\sigma},z'-z)\exp[-\tfrac{1}{2}i\vec{\sigma}\cdot\vec{\vartheta}(z'+z)]$$

$$\times\{\exp[-i\mu(z'-z)]+\exp[-i\mu(z'+z)]$$

$$+\exp[i\mu(z'+z)]+\exp[i\mu(z'-z)]\}, \qquad (3.59)$$

and

$$F_{\ell\phi}(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R\!\!\int_0^R dz' \, dz \; F_N(\vec{\sigma}, z'-z) \exp[-\tfrac{1}{2} i \vec{\sigma}\cdot\vec{\vartheta}(z'+z)]$$

$$\times \{ \exp[-i\mu(z'-z)] + \exp[-i\mu(z'+z)] - \exp[i\mu(z'+z)] - \exp[i\mu(z'-z)] \}, \qquad (3.60)$$

where

$$\mu = \tfrac{1}{8} k\vartheta^2 + \tfrac{1}{2}\sigma^2/k \qquad . \qquad (3.61)$$

At this point we shall introduce new variables of integration, namely

$$u = z' - z \qquad , \qquad (3.62)$$

and

$$v = \tfrac{1}{2}(z' + z) \qquad . \qquad (3.63)$$

Because of the finite range of the refractive-index correlation function (which range we assume is much less than the propagation path length R) we can consider the limits of the u-integration to run from $-\infty$ to $+\infty$, while the limits of the v integration run from 0 to R. Thus we can rewrite Eq.'s (58), (59), and (60) as

$$F_\ell(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R dv \int du \; F_N(\vec{\sigma}, u) \exp(-i\vec{\sigma}\cdot\vec{\vartheta}\, v)$$

$$\times [\exp(-i\mu u) - \exp(-2i\mu v) - \exp(2i\mu v) + \exp(i\mu u)], \qquad (3.64)$$

$$F_\phi(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R dv \int du \; F_N(\vec{\sigma}, u) \exp(-i\vec{\sigma}\cdot\vec{\vartheta}\, v)$$

$$\times [\exp(-i\mu u) + \exp(-2i\mu v) + \exp(2i\mu v) + \exp(i\mu u)], \qquad (3.65)$$

and

$$F_{\ell\phi}(\vec{\sigma},\vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R dv \int du \; F_N(\vec{\sigma}, u) \exp(-i\vec{\sigma}\cdot\vec{\vartheta}\, v)$$

$$\times [\exp(-i\mu u) + \exp(-2i\mu v) - \exp(2i\mu v) - \exp(i\mu u)] \; . \qquad (3.66)$$

Rather than carry the $F_{\ell\phi}$ - term any further we shall now show that it is identically equal to zero. This follows from the fact that since ℓ and ϕ are each real valued (random) quantities, then $C_{\ell\phi}$ must also be real. From this fact, together with Eq. (10), we can see that

$$F^*_{\ell\phi}\ (-\vec{\sigma}, \vec{\vartheta}) = F_{\ell\phi}(\vec{\sigma}, \vec{\vartheta}) \qquad . \qquad (3.67)$$

Similarly since the random refractive-index, n, and its covariance function, C_N, are real valued, it follows from consideration of Eq. (11) that

$$F^*_N\ (-\vec{\sigma}, \xi) = F_N(\vec{\sigma}, \xi) \qquad . \qquad (3.68)$$

We note that according to Eq. (61) the value of μ is unchanged when $\vec{\sigma}$ is replaced by $-\vec{\sigma}$. This means that we can write as an alternative form of Eq. (66) the equation

$$F^*_{\ell\phi}\ (-\vec{\sigma}, \vec{\vartheta}) = (\tfrac{1}{2}k)^2 \int_0^R dv \int du\ F^*_N(-\vec{\sigma}, u) \exp(-i\vec{\sigma}\cdot\vec{\vartheta}v)$$

$$\times [\exp(i\mu u) + \exp(2i\mu v) - \exp(-2i\mu v) - \exp(-i\mu u)] .$$

$$(3.69)$$

Making use of Eq. (68) we can recast this expression in the form

$$F^*_{\ell\phi}\ (-\vec{\sigma}, \vec{\vartheta}) = -(\tfrac{1}{2}k)^2 \int_0^R dv \int du\ F_N(\vec{\sigma}, u) \exp(-i\vec{\sigma}\cdot\vec{\vartheta}v)$$

$$\times [\exp(-i\mu u) + \exp(-2i\mu v) - \exp(2i\mu v) - \exp(i\mu u)] .$$

$$(3.70)$$

It is obvious that Eq. 's (66) and (70) are compatible with Eq. (67) only if

$$F_{\ell\phi}\ (\vec{\sigma}, \vec{\vartheta}) \equiv 0 \qquad . \qquad (3.71)$$

According to Eq. (50) this means that

$$C_{\ell\phi}\ (\vec{\rho}, \vec{\vartheta}) \equiv 0 \qquad . \qquad (3.72)$$

We are now ready to return to the evaluation of F_ℓ and F_ϕ and then of C_ℓ and C_ϕ .

We can carry out the u-integration in Eq.'s (64) and (65) by noting that in accordance with Eq. (13) the u-integrations can be considered to correspond to taking a Fourier transformation along the z-axis. This will convert the two dimensional Fourier transform $F_N(\vec{\sigma}, u)$ to the three-dimensional Fourier transform $\mathcal{F}_N(\vec{\sigma}, s)$, where s will take a value of either 0 or $\pm 2\mu$. We argue that μ is (except for some very extreme conditions of no interest to us) much smaller than the magnitude of $\vec{\sigma}$, so that in accordance with Eq. (14) we can consider the three-dimensional Fourier transform in all cases to have arguments of $(\vec{\sigma}, 0)$. It is convenient to write this as $\mathcal{F}_N(\sigma)$ where

$$\mathcal{F}_N(\sigma) \equiv \mathcal{F}_N(\vec{\sigma}, 0) \qquad , \qquad (3.73)$$

and according to Eq. (14)

$$\mathcal{F}_N(\sigma) = 8.16 \; C_N^2 \; \sigma^{-11/3} \qquad . \qquad (3.74)$$

When we carry out the u-integrations in Eq.'s (64) and (65) we get

$$F_{\ell}(\vec{\sigma}, \vec{\vartheta}) = \tfrac{1}{2} k^2 \int_0^R dv \; \mathcal{F}_N(\sigma) \exp(-i\vec{\sigma} \cdot \vec{\vartheta} v) \left[1 - \cos(2\mu v)\right] , \qquad (3.75)$$

and

$$F_{\phi}(\vec{\sigma}, \vec{\vartheta}) = \tfrac{1}{2} k^2 \int_0^R dv \; \mathcal{F}_N(\sigma) \exp(-i\vec{\sigma} \cdot \vec{\vartheta} v) \left[1 + \cos(2\mu v)\right] . \qquad (3.76)$$

Now substituting these results into Eq.'s (48) and (49) we obtain the equation

$$C_{\ell}(\vec{\rho}, \vec{\vartheta}) = \frac{8.16}{8} (k/\pi)^2 \int_0^R dv \; C_N^2 \int d\vec{\sigma} \; \sigma^{-11/3} \left[1 - \cos(2\mu v)\right]$$

$$\times \exp\left[i\vec{\sigma} \cdot (\vec{\rho} - \vec{\vartheta} v)\right] \qquad , \qquad (3.77)$$

and

$$C_\phi(\vec{\rho}, \vec{\vartheta}) = \frac{8.16}{8}(k/\pi)^2 \int_0^R dv\ C_N^2 \int d\vec{\sigma}\ \sigma^{-11/3}\left[1+\cos(2\mu v)\right]$$

$$\times \exp\left[i\,\vec{\sigma}\cdot(\vec{\rho}-\vec{\vartheta}v)\right] \quad . \qquad (3.78)$$

Making use of the well known formula that

$$\int d\vec{\sigma}\ g(\sigma)\exp(i\alpha\,\vec{\sigma}\cdot\vec{x}) = 2\pi\int_0^\infty \sigma d\sigma\ g(\sigma)\ J_0(\alpha\sigma x)\ , \qquad (3.79)$$

where $g(\sigma)$ is any function, then we can rewrite Eq. 's (77) and (78) as

$$C_\ell(\vec{\rho}, \vec{\vartheta}) = \frac{8.16}{4\pi}\ k^2 \int_0^R dv\ C_N^2 \int_0^\infty d\sigma\ \sigma^{-8/3}\ J_0(\sigma\,|\vec{\rho}-\vec{\vartheta}v|)$$

$$\times\left[1-\cos(2\mu v)\right] \quad , \qquad (3.80)$$

and

$$C_\phi(\vec{\rho}, \vec{\vartheta}) = \frac{8.16}{4\pi}\ k^2 \int_0^R dv\ C_N^2 \int_0^\infty d\sigma\ \sigma^{-8/3}\ J_0(\sigma\,|\vec{\rho}-\vec{\vartheta}v|)$$

$$\times\left[1+\cos(2\mu v)\right] \quad . \qquad (3.81)$$

These two equations, together with Eq. (72), represent our basic results for infinite plane wave propagation. In the next subsection we shall present heuristic arguments for the changes needed to make these results apply for point source/spherical wave propagation.

1.3.4 Point Source/Spherical Wave Propagation

Without going through an even more extensive derivation than that of the preceding subsection we can infer the appropriate spherical wave results from our infinite plane wave results. To do this we start with Eq. 's (80) and (81) [as well as Eq. (72)] for the infinite plane wave case, and apply the following heuristic argument. The significant difference between the infinite plane wave case and the spherical wave case is that for the infinite plane wave case $\vec{\rho}$ not only denotes the separation of two points on the measurement (aperture) plane but also denotes the separation of two rays

traveling to those two points quite independent of where along the v-integration (propagation) path we examine the matter. On the other hand for the spherical wave case the separation between the two rays is $\vec{\rho}\,(R-v)/R$, which varies with v along the integration (propagation) path. We argue that the presence of $\vec{\rho}$ inside the v-integrand in our infinite plane wave results is supposed to correspond to the separation of the two rays leading to the measurement points as a function of v. Accordingly we would write for the spherical wave case

$$C_{\ell}\,(\vec{\rho},\vec{\vartheta}) \;=\; \frac{8.16}{4\pi}\,k^2 \int_0^R dv\;C_N{}^2 \int_0^\infty d\sigma\;\;\sigma^{-8/3}\,J_0\left(\sigma\left|\vec{\rho}\;\frac{R-v}{R} - \vec{\vartheta}v\right|\right)$$

$$\times\,[1-\cos\,(2\mu v)] \qquad , \qquad (3.82)$$

and

$$C_{\phi}\,(\vec{\rho},\vec{\vartheta}) \;=\; \frac{8.16}{4\pi}\,k^2 \int_0^R dv\;C_N{}^2 \int_0^\infty d\sigma\;\sigma^{-8/3}\,J_0\left(\sigma\left|\vec{\rho}\;\frac{R-v}{R} - \vec{\vartheta}v\right|\right)$$

$$\times\,[1+\cos\,(2\mu v)] \qquad . \qquad (3.83)$$

And of course, we still have the result that

$$C_{\ell\phi}\,(\vec{\rho},\vec{\vartheta}) \;=\; 0 \qquad . \qquad (3.84)$$

It is convenient to introduce the notation

$$\vec{\rho}_v \;=\; \vec{\rho}\;(R-v)/R \qquad , \qquad (3.85)$$

and

$$\vec{\vartheta}_v \;=\; \vec{\vartheta}\,v \qquad , \qquad (3.86)$$

so that we can rewrite Eq.'s (82) and (83) as

$$C_{\ell}(\vec{\rho},\vec{\vartheta}) \;=\; \frac{8.16}{4\pi}\,k^2 \int_0^R dv\;\;C_N{}^2 \int_0^\infty d\sigma\;\;\sigma^{-8/3}\,J_0\,(\sigma|\vec{\rho}_v - \vec{\vartheta}_v|)$$

$$\times\,[\,1-\cos(2\mu v)\,] \qquad , \qquad (3.87)$$

and

$$C_{\phi}(\vec{\rho}, \vec{\vartheta}) = \frac{8.16}{4\pi} k^2 \int_0^R dv \; C_N^2 \int_0^\infty d\sigma \; \sigma^{-8/3} J_0(\sigma |\vec{\rho}_V - \vec{\vartheta}_V|)$$

$$\times [1 + \cos(2\mu u)] \qquad . \qquad (3.88)$$

With these spherical wave propagation theory results in hand we are now ready to return to consideration of adaptive optics systems performance. At this point we are prepared to undertake the problem of evaluating anisoplanatism effects, i. e., treating the case when the target direction, $\vec{\theta}_T$, and the direction to the beacon, $\vec{\theta}_B$, are distinct. We treat this in the next section.

1.4 System Performance With Anisoplanatsim

In Section 1.2 we developed all of the basic formulations we need to study anisoplanatism effects on adaptive optics imaging and laser transmitter systems, with and without random apodization correction. It will prove convenient to gather the relevant formulas together at this point.

For the adaptive optics imaging system without random apodization correction the relevant formulas are Eq.'s (2.41), (2.42), and (2.44). We repeat them here as

$$\langle \mathcal{I}_{IW/O}(\vec{f}) \rangle = \tfrac{1}{2} \langle B_{IW/O} \rangle \int d\vec{r} \, \langle U^*_{IW/O}(\vec{r} + \lambda \vec{f}) U_{IW/O}(\vec{r}) \rangle \cdot \tag{4.1}$$

$$\langle B_{IW/O} \rangle = \{ \int d\vec{r} \, \tfrac{1}{2} \langle U^*_{IW/O}(\vec{r}) U_{IW/O}(\vec{r}) \rangle \}^{-1} \quad . \tag{4.2}$$

and

$$U_{IW/O}(\vec{r}) = A W(\vec{r}) \exp \{ ik [R(1 + \tfrac{1}{2}\theta_T^2) - \vec{r} \cdot \vec{\theta}_T] + i [\phi(\vec{r}, \vec{\theta}_T) - \phi(\vec{r}, \vec{\theta}_B)] + \ell(\vec{r}, \vec{\theta}_T) \} \quad . \tag{4.3}$$

For the adaptive optics imaging system with random apodization compensation the relevant formulas are Eq.'s (2.59), (2.60), and (2.62). We repeat them here as

$$\langle \mathcal{I}_{IW}(\vec{f}) \rangle = \tfrac{1}{2} \langle B_{IW} \rangle \int d\vec{r} \, \langle U^*_{IW}(\vec{r} + \lambda \vec{f}) U_{IW}(\vec{r}) \rangle , \tag{4.4}$$

$$\langle B_{IW} \rangle = \{ \int d\vec{r} \, \tfrac{1}{2} \langle U^*_{IW}(\vec{r}) U_{IW}(\vec{r}) \rangle \}^{-1} \tag{4.5}$$

and

$$U_{IW}(\vec{r}) = A W(\vec{r}) \exp \{ ik [R(1 + \tfrac{1}{2}\theta_T^2) - \vec{r} \cdot \vec{\theta}_T] + i [\phi(\vec{r}, \vec{\theta}_T) - \phi(\vec{r}, \vec{\theta}_B)] + [\ell(\vec{r}, \vec{\theta}_T) - \ell(\vec{r}, \vec{\theta}_B)] \} \cdot \tag{4.6}$$

For the adaptive optics laser transmitter without random apodization correction the relevant formulas are Eq.'s (2.68), (2.69), (2.70), and (2.72), and (2.73). We repeat them here as

$$\langle G_{LTW/O} \rangle = R^2 \langle \mathscr{P}_{LTW/O} \rangle / \langle P_{LTW/O} \rangle \quad , \qquad (4.7)$$

$$\langle \mathscr{P}_{LTW/O} \rangle = \tfrac{1}{2} \langle u^*_{LTW/O} \; u_{LTW/O} \rangle \quad , \qquad (4.8)$$

$$u_{LTW/O} = -\frac{i}{\lambda R} \exp\left[ikR(1+\tfrac{1}{2}\vec{\theta}_T{}^2)\right] A \int d\vec{r}\, W(\vec{r}) \times \exp\{i[\phi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)] + \ell(\vec{r},\vec{\theta}_T)\} \, , \qquad (4.9)$$

$$\langle P_{LTW/O} \rangle = \int d\vec{r}\, \tfrac{1}{2} \langle U^*_{LTW/O}(\vec{r}) \; U_{LTW/O}(\vec{r}) \rangle \quad , \qquad (4.10)$$

and

$$U_{LTW/O}(\vec{r}) = A\, W(\vec{r}) \exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}_T] - i\phi(\vec{r},\vec{\theta}_B)\} . \qquad (4.11)$$

For the adaptive optics laser transmitter with random apodization correction the relevant formulas are Eq.'s (2.87), (2.88), (2.89), (2.90), and (2.105). We repeat them here as

$$\langle G_{LTW} \rangle = R^2 \langle \mathscr{P}_{LTW} \rangle / \langle P_{LTW} \rangle \quad , \qquad (4.12)$$

$$\langle \mathscr{P}_{LTW} \rangle = \tfrac{1}{2} \langle u^*_{LTW} \; u_{LTW} \rangle \quad , \qquad (4.13)$$

$$u_{LTW} = -\frac{i}{\lambda R} \exp\left[ikR(1+\tfrac{1}{2}\theta_T{}^2)\right] A \int d\vec{r}\; W(\vec{r}) \times \exp\{i[\phi(\vec{r},\vec{\theta}_T) - \phi(\vec{r},\vec{\theta}_B)] + [\ell(\vec{r},\vec{\theta}_T) + \ell(\vec{r},\vec{\theta}_B)]\}, \qquad (4.14)$$

$$\langle P_{LTW} \rangle = \int d\vec{r}\; \tfrac{1}{2} \langle U^*_{LTW}(\vec{r}) \; U_{LTW}(\vec{r}) \rangle \quad , \qquad (4.15)$$

and

$$U_{LTW}(\vec{r}) = A\, W(\vec{r}) \exp\{ik[O(\vec{r}) + \vec{r}\cdot\vec{\theta}_T] - i\phi(\vec{r},\vec{\theta}_B) + \ell(\vec{r},\vec{\theta}_B)\} . \qquad (4.16)$$

The function $O(\vec{r})$ used in Eq.'s (11) and (16) is a real valued function, defined in Eq. (2.4). It will be noted that Eq. (14) is not an exact copy of Eq. (2.89). Rather it has been obtained from Eq. (2.89) by substituting Eq.'s (2.105), (2.3), and (2.4) into (2.89), and simplifying.

With these formulas in hand we are ready to start a detailed evaluation

of the expected performance of each of the four systems under conditions of anisoplanatism. However, before we take up those four tasks separately it will be convenient to develop some relevant statistical results. We shall do that in the next subsection.

1.4.1 Exponential Statistical Functions Associated With Anisoplanatism

In examining the preceding sixteen equations we can see that certain exponential functions of the random phase and log-amplitude will have to have their ensemble average values determined. These can be written as

$$E_{IW/O}(\vec{\rho}) = \langle \exp\{-i[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T) - \phi(\vec{r}+\vec{\rho},\vec{\theta}_B) - (\vec{r},\vec{\theta}_T) + \phi(\vec{r},\vec{\theta}_B)] + [\ell(\vec{r}+\vec{\rho},\vec{\theta}_T) + \ell(\vec{r},\vec{\theta}_T)]\}\rangle \quad , \qquad (4.17)$$

$$E_{IW}(\vec{\rho}) = \langle \exp\{-i[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T) - \phi(\vec{r}+\vec{\rho},\vec{\theta}_B) - \phi(\vec{r},\vec{\theta}_T) + \phi(\vec{r},\vec{\theta}_B)] + [\ell(\vec{r}+\vec{\rho},\vec{\theta}_T) - \ell(\vec{r}+\vec{\rho},\vec{\theta}_B) + \ell(\vec{r},\vec{\theta}_T) - \ell(\vec{r},\vec{\theta}_B)]\}\rangle \; , \qquad (4.18)$$

$$E_{LTW/O}(\vec{\rho}) = \langle \exp\{-i[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T) - \phi(\vec{r}+\vec{\rho},\vec{\theta}_B) - \phi(\vec{r},\vec{\theta}_T) + \phi(\vec{r},\vec{\theta}_B)] + [\ell(\vec{r}+\vec{\rho},\vec{\theta}_T) + \ell(\vec{r},\vec{\theta}_T)]\}\rangle \quad , \qquad (4.19)$$

$$E_{LTW1}(\vec{\rho}) = \langle \exp\{-i[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T) - \phi(\vec{r}+\vec{\rho},\vec{\theta}_B) - \phi(\vec{r},\vec{\theta}_T) + \phi(\vec{r},\vec{\theta}_B)] + [\ell(\vec{r}+\vec{\rho},\vec{\theta}_T) + \ell(\vec{r}+\vec{\rho},\vec{\theta}_B) + \ell(\vec{r},\vec{\theta}_T) + \ell(\vec{r},\vec{\theta}_B)]\}\rangle , \qquad (4.20)$$

and

$$E_{LTW2} = \langle \exp[2\,\ell(\vec{r},\vec{\theta}_B)]\rangle \quad . \qquad (4.21)$$

The key to the evaluation of these expressions lies, of course, in the application of Eq. (2.35), and then in the use of Eq.'s (3.84), (3.87), and (3.88).

Making use of Eq. (2.35), and relying on Eq. (3.84) to allow us to drop from our formulation all terms involving the ensemble average of the

product of phase and log amplitude, we can write

$$E_{IW/O}(\vec{\rho}) = \exp\{-\tfrac{1}{2}\langle[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T)-\phi(\vec{r}+\vec{\rho},\vec{\theta}_B)-\phi(\vec{r},\vec{\theta}_T)+\phi(\vec{r},\vec{\theta}_B)]^2\rangle$$

$$+2\bar{\ell}+\tfrac{1}{2}\langle\{[\ell(\vec{r}+\vec{\rho},\vec{\theta}_T)-\ell]+[\ell(\vec{r},\vec{\theta}_T)-\ell]\}^2\rangle\} . \quad (4.22)$$

Making use of Eq.'s (2.37), (3.4), and (3.5) and with

$$\vec{\vartheta} = \vec{\theta}_T - \vec{\theta}_B \quad , \quad (4.23)$$

we can develop from Eq. (22) the result that

$$E_{IW/O}(\vec{\rho}) = \exp[-2C_\phi(0,0)+2C_\phi(\vec{\rho},0)+2C_\phi(0,\vec{\vartheta})-C_\phi(\vec{\rho},\vec{\vartheta})$$

$$-C_\phi(\vec{\rho},-\vec{\vartheta})-C_\ell(0,0)+C_\ell(\vec{\rho},0)] . \quad (4.24)$$

Making use of Eq.'s (3.61), (3.82), and (3.83) we can rewrite this as

$$E_{IW/O}(\vec{\rho}) = \exp[S_{IW/O}(\vec{\rho})] \quad , \quad (4.25)$$

where

$$S_{IW/O}(\vec{\rho}) = [C_\ell(0,0)-C_\ell(\vec{\rho},0)]+\frac{8.16}{4\pi}k^2\int_0^R dv\ C_N^2\int_0^\infty d\sigma\,\sigma^{-8/3}$$

$$\times \Big\{2[1+\cos(\tfrac{\sigma^2}{k}v)][-1+J_0(\sigma\rho_v)]$$

$$+\{1+\cos[(\tfrac{1}{4}k\vartheta^2+\tfrac{\sigma^2}{k})v]\}[2J_0(\sigma\vartheta_v)-J_0(\sigma|\vec{\rho}_v-\vec{\vartheta}_v|)-J_0(\sigma|\vec{\rho}_v+\vec{\vartheta}_v|)]$$

$$+2[1-\cos(\tfrac{\sigma^2}{k}v)][-1+J_0(\sigma\rho_v)]\Big\} . \quad (4.26)$$

To proceed beyond this point we have to make the approximation that the turbulence processes are "in the near field" so that we can approximate the ϑ-dependent $[1+\cos(\ldots)]$-term by 2. When we do this and restore the log-amplitude covariance dependencies to explicit form, we get

$$S_{IW/O}(\vec{\rho}) = [C_\ell(0,0)-C_\ell(\vec{\rho},0)]+\frac{8.16}{\pi}k^2\int_0^R dv\ C_N^2\int_0^\infty d\sigma\,\sigma^{-8/3}$$

$$\times[-1+J_0(\sigma\rho_v)+J_0(\sigma\vartheta_v)-\tfrac{1}{2}J_0(\sigma|\vec{\rho}_v+\vec{\vartheta}_v|)-\tfrac{1}{2}J_0(\sigma|\vec{\rho}_v-\vec{\vartheta}_v|)] .$$

(4.27)

To evaluate the σ - integral we make use of the equation[10]

$$\int dx\ x^{\mu} J_{\nu}(a x) = 2^{\mu} a^{-\mu-1}\ \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu),$$

$$\text{for } \mathrm{Re}(\nu-1) < \mathrm{Re}(\mu) < \tfrac{1}{2}, \text{ and } a > 0. \quad (4.28)$$

Strickly speaking this formula is not applicable to our problem since it would involve $\mu = 8/3$ and $\nu = 0$, for which the conditions of applicability are violated. However, by arguing that the -1 term can be considered to be $-J_0(ax)$ with $a \approx 0$, this formula can be made applicable to the entire integral of Eq. (27) by arguments of analytic continuation in μ*. Thus we obtain the result that

$$S_{IW/O}(\vec{\rho}) = [\,C_{\ell}(0,0) - C_{\ell}(\vec{\rho},0)] - 2.905\, k^2 \int_0^R dv\, C_N^2$$

$$\times[\rho_v{}^{5/3} + \vartheta_v{}^{5/3} - \tfrac{1}{2}|\vec{\rho}_v + \vec{\vartheta}_v|^{5/3} - \tfrac{1}{2}|\vec{\rho}_v - \vec{\vartheta}_v|^{5/3}]. \quad (4.29)$$

Proceeding in essentially the same way for the evaluation of $E_{IW}(\vec{\rho})$ we write

$$E_{IW}(\vec{\rho}) = \exp\{-\tfrac{1}{2}\langle[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T) - \phi(\vec{r}+\vec{\rho},\vec{\theta}_B) - \phi(\vec{r},\vec{\theta}_T) + \phi(\vec{r},\vec{\theta}_B)]^2\rangle$$

$$+\tfrac{1}{2}\langle[\ell(\vec{r}+\vec{\rho},\vec{\theta}_T) - \ell(\vec{r}+\vec{\rho},\vec{\theta}_B) + \ell(\vec{r},\vec{\theta}_T) - \ell(\vec{r},\vec{\theta}_B)]^2\rangle\}. \quad (4.30)$$

This can be recast in the form

$$E_{IW}(\vec{\rho}) = \exp[-2C_{\phi}(0,0) + 2C_{\phi}(\vec{\rho},0) + 2C_{\phi}(0,\vec{\vartheta}) - C_{\phi}(\vec{\rho},\vec{\vartheta}) - C_{\phi}(\vec{\rho},-\vec{\vartheta})$$

$$+ 2C_{\ell}(0,0) + 2C_{\ell}(\vec{\rho},0) - 2C_{\ell}(0,\vec{\vartheta}) - C_{\ell}(\vec{\rho},\vec{\vartheta}) - C_{\ell}(\vec{\rho},-\vec{\vartheta})].$$

$$(4.31)$$

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* The analytic continuation argument is basically as follows. Replace the -8/3 power in Eq. (27) by μ and then use Eq. (28) to evaluate this integral. The result is an analytic function of μ which is clearly valid for $\mu > -1$. But since the quantity in the square brackets in Eq. (27) is proportional to σ^2 for small values of σ, the integral does not diverge for any value of μ greater than -3. Accordingly our analytic function of μ representing the value of the entire integral is valid, by analytic continuation at least to $\mu = -8/3$.

We can rewrite this as

$$E_{IW}(\vec{\rho}) = \exp[S_{IW}(\vec{\rho})] \quad , \qquad (4.32)$$

where

$$S_{IW}(\vec{\rho}) = 4[C_\ell(0,0) - C_\ell(0,\vec{\vartheta})] + \frac{8.16}{4\pi}k^2\int_0^R dv\ C_N^2\int_0^\infty d\sigma\ \sigma^{-8/3}$$

$$\times \Big\langle\!\Big\langle 2[1+\cos(\frac{\sigma^2}{k}v)][-1+J_0(\sigma\rho_v)]$$

$$+\{1+\cos[(\tfrac{1}{4}k\vartheta^2+\frac{\sigma^2}{k})v]\}[2J_0(\sigma\vartheta_v)-J_0(\sigma|\vec{\rho}_v-\vec{\vartheta}_v|)-J_0(\sigma|\vec{\rho}_v+\vec{\vartheta}_v|)]$$

$$+2[1-\cos(\frac{\sigma^2}{k}v)][-1+J_0(\sigma\rho_v)]$$

$$+\{1-\cos[(\tfrac{1}{4}k\vartheta^2+\frac{\sigma^2}{k})v]\}[2J_0(\sigma\vartheta_v)-J_0(\sigma|\vec{\rho}_v-\vec{\vartheta}_v|)-J_0(\sigma|\vec{\rho}_v+\vec{\vartheta}_v|)]\Big\rangle\!\Big\rangle$$

$$= 4[C_\ell(0,0)-C_\ell(0,\vec{\vartheta})] + \frac{8.16}{\pi}k^2\int_0^R dv\ C_N^2\int_0^\infty d\sigma\ \sigma^{-8/3}$$

$$\times[-1+J_0(\sigma\rho_v)+J_0(\sigma\vartheta_v)-\tfrac{1}{2}J_0(\sigma|\vec{\rho}_v-\vec{\vartheta}_v|)-\tfrac{1}{2}J_0(\sigma|\vec{\rho}_v+\vec{\vartheta}_v|)] \ . \qquad (4.33)$$

We can carry out the σ-integration exactly as we did before. Doing so we obtain the result that

$$S_{IW}(\vec{\rho}) = 4[C_\ell(0,0) - C_\ell(0,\vec{\vartheta})] - 2.905\,k^2\int_0^R dv\ C_N^2$$

$$\times[\rho_v^{5/3}+\vartheta_v^{5/3}-\tfrac{1}{2}|\vec{\rho}_v+\vec{\vartheta}_v|^{5/3}-\tfrac{1}{2}|\vec{\rho}_v-\vec{\vartheta}_v|^{5/3}] \ . \qquad (4.34)$$

To evaluate $E_{LT\,W/O}(\vec{\rho})$ we note from a comparison of Eq.'s (17) and (19) that the expression for $E_{LT\,W/O}(\vec{\rho})$ is identical to that for $E_{IW/O}(\vec{\rho})$. Accordingly we can simply copy Eq.'s (25) and (29) giving

$$E_{LT\,W/O}(\vec{\rho}) = \exp[S_{LT\,W/O}(\vec{\rho})] \quad , \qquad (4.35)$$

and

$$S_{LTWO}(\vec{\rho}) = [C_\ell(0,0) - C_\ell(\vec{\rho},0)] - 2.905\,k^2 \int_0^R dv\; C_N^2$$

$$\times[\rho_v^{5/3} + \vartheta_v^{5/3} - \tfrac{1}{2}|\vec{\rho}_v + \vec{\vartheta}_v|^{5/3} - \tfrac{1}{2}|\vec{\rho}_v - \vec{\vartheta}_v|^{5/3}]\,. \qquad (4.36)$$

To evaluate $E_{LTW1}(\vec{\rho})$ we start by writing

$$E_{LTW1}(\vec{\rho}) = \exp\left\langle\!\!\left\langle -\tfrac{1}{2}\langle[\phi(\vec{r}+\vec{\rho},\vec{\theta}_T) - \phi(\vec{r}+\vec{\rho},\vec{\theta}_B) - \phi(\vec{r},\vec{\theta}_T) + \phi(\vec{r},\vec{\theta}_B)]^2\rangle \right.\right.$$

$$+4\bar{\ell} + \tfrac{1}{2}\langle\{[\ell(\vec{r}+\vec{\rho},\vec{\theta}_T) - \bar{\ell}] + [\ell(\vec{r}+\vec{\rho},\vec{\theta}_B) - \bar{\ell}] + [\ell(\vec{r},\vec{\theta}_T) - \bar{\ell}]$$

$$\left.\left. + [\ell(\vec{r},\vec{\theta}_B) - \bar{\ell}]\}^2\rangle \right\rangle\!\!\right\rangle \qquad . \qquad (4.37)$$

We can rewrite this as

$$E_{LTW1}(\vec{\rho}) = \exp[-2C_\phi(0,0) + 2C_\phi(\vec{\rho},0) + 2C_\phi(0,\vec{\vartheta}) - C_\phi(\vec{\rho},\vec{\vartheta}) - C_\phi(\vec{\rho},-\vec{\vartheta})$$

$$-2C_\ell(0,0) + 2C_\ell(\vec{\rho},0) + 2C_\ell(0,\vec{\vartheta}) + C_\ell(\vec{\rho},\vec{\vartheta}) + C_\ell(\vec{\rho},-\vec{\vartheta})]. \quad (4.38)$$

Again making use of Eq.'s (3.82) and (3.83) we can rewrite this as

$$E_{LTW1}(\vec{\rho}) = \exp[S_{LTW1}(\vec{\rho})] \qquad , \qquad (4.39)$$

where

$$S_{LTW1}(\vec{\rho}) = \frac{8.16}{4\pi}k^2 \int_0^R dv\; C_N^2 \int_0^\infty d\sigma\, \sigma^{-8/3}$$

$$\times\left\langle\!\!\left\langle 2[1+\cos(\tfrac{\sigma^2}{k}v)][-1+J_0(\sigma\rho_v)] \right.\right.$$

$$+\{1+\cos[(\tfrac{1}{4}k\vartheta^2 + \tfrac{\sigma^2}{k})v]\}[2J_0(\sigma\vartheta_v) - J_0(\sigma|\vec{\rho}_v - \vec{\vartheta}_v|) - J_0(\sigma|\vec{\rho}_v + \vec{\vartheta}_v|)]$$

$$+2[1-\cos(\tfrac{\sigma^2}{k}v)][-1+J_0(\sigma\rho_v)]$$

$$\left.\left. +\{1-\cos[(\tfrac{1}{4}k\vartheta^2 + \tfrac{\sigma^2}{k})v]\}[2J_0(\sigma\vartheta_v) + J_0(\sigma|\vec{\rho}_v - \vec{\vartheta}_v|) + J_0(\sigma|\vec{\rho}_v + \vec{\vartheta}_v|)] \right\rangle\!\!\right\rangle$$

$$= \frac{8.16}{\pi}k^2 \int_0^R dv\, C_N^2 \int_0^\infty d\sigma\, \sigma^{-8/3}\{[-1+J_0(\sigma\rho_v) + J_0(\sigma\vartheta_v)]$$

$$-\tfrac{1}{2}\cos[(\tfrac{1}{4}k\vartheta^2 + \tfrac{\sigma^2}{k})v][J_0(\sigma|\vec{\rho}_v - \vec{\vartheta}_v|) + J_0(\sigma|\vec{\rho}_v + \vec{\vartheta}_v|)]\}\,. \quad (4.40)$$

Here again we make the same approximation as before, i. e., that we are dealing with a "near field" problem, so that the cosine-term can be approximated by unity. This allow us to write

$$S_{LTW1}(\vec{\rho}) = \frac{8.16}{\pi} k^2 \int_0^R dv \; C_N^2 \int_0^\infty d\sigma \, \sigma^{-8/3}$$

$$\times \left[-1 + J_0(\sigma \rho_v) + J_0(\sigma \vartheta_v) - \tfrac{1}{2} J_0(\sigma |\vec{\rho}_v - \vec{\vartheta}_v|) - \tfrac{1}{2} J_0(\sigma |\vec{\rho}_v + \vec{\vartheta}_v|) \right] . \quad (4.41)$$

In obtaining Eq. (29) we carried out the same σ-integration that we have to perform here. Applying that result we can now write

$$S_{LTW1}(\vec{\rho}) = -2.905 k^2 \int_0^R dv \; C_N^2 (\rho_v^{5/3} + \vartheta_v^{5/3} - \tfrac{1}{2} |\vec{\rho}_v + \vec{\vartheta}_v|^{5/3} - \tfrac{1}{2} |\vec{\rho}_v - \vec{\vartheta}_v|^{5/3}) . \quad (4.42)$$

With this result in hand we now turn our attention to the evaluation of E_{LTW2}, as defined in Eq. (21). But this quantity was (implicitly) evaluated in going from Eq. (2.45) to Eq. (2.49). From that work we can see that

$$E_{LTW2} = 1 \quad . \quad (4.43)$$

With these results in hand we are ready to take up the question of developing tractable expressions for system performance for each of the four types of systems we have been considering. This is treated in the next subsection.

1.4.2 System Performance Formulas

Before starting the presentation of results in this section it will be convenient to introduce the following function which we will find is central to the anisoplanatism effect for all cases considered. We define the basic anisoplanatism function as

$$S(\vec{\rho}, \vec{\vartheta}) = 2.905 \, k^2 \int_0^R dv \; C_N^2 (\rho_v^{5/3} + \vartheta_v^{5/3} - \tfrac{1}{2} |\vec{\rho}_v + \vec{\vartheta}_v|^{5/3} - \tfrac{1}{2} |\vec{\rho}_v - \vec{\vartheta}_v|^{5/3}) , \quad (4.44)$$

where we recall that

$$\vec{\rho}_v = \vec{\rho} \, (R - v)/R \quad , \quad (4.45)$$

$$\vec{\vartheta}_v = \vec{\vartheta} \, v \quad . \quad (4.46)$$

To evaluate the performance of an adaptive optics imaging system without random apodization compensation we use Eq.'s (1), (2), (3), (17), (25), and (29). It is easy to see that when Eq. (3) is twice substituted into Eq. (2) we get

$$\langle B_{I\,W/O} \rangle = \{ \int d\vec{r}\, \tfrac{1}{2} A^* A \;\; W(\vec{r})\, E_{I\,W/O}(0) \}^{-1}$$

$$= \{ \tfrac{1}{2} A^* A \,(\tfrac{1}{4} \pi D^2)\;\; E_{I\,W/O}(0) \}^{-1} \qquad . \qquad (4.47)$$

From consideration of Eq.'s (25) and (29) it is obvious that

$$E_{I\,W/O}(0) = 1 \qquad , \qquad (4.48)$$

so that

$$\langle B_{I\,W/O} \rangle = \{ \tfrac{1}{2} A^* A \,(\tfrac{1}{4} \pi D^2) \}^{-1} \qquad . \qquad (4.49)$$

When Eq. (3) is twice substituted into Eq. (1) we get

$$\langle \mathcal{I}_{I\,W/O}(\vec{f}) \rangle = \tfrac{1}{2} \langle B_{I\,W/O} \rangle \int d\vec{r}\; A^* A \;\; W(\vec{r}+\lambda\vec{f})\; W(\vec{r})\; E_{I\,W/O}(\lambda\vec{f})$$

$$= \tfrac{1}{2} A^* A \langle B_{I\,W/O} \rangle\; E_{I\,W/O}(\lambda\vec{f}) \int d\vec{r}\;\; W(\vec{r}+\lambda\vec{f})\; W(\vec{r})$$

$$= \tfrac{1}{2} A^* A \langle B_{I\,W/O} \rangle\; E_{I\,W/O}(\lambda\vec{f})\; (\tfrac{1}{4} \pi D^2)\, K(\lambda\vec{f}) \quad . \qquad (4.50)$$

Now making use of Eq.'s (2.22), (49), (44), (25), and (29) we can write this as

$$\langle \mathcal{I}_{I\,W/O}(\vec{f}) \rangle = \mathcal{I}_{DL}(\vec{f}) \exp[\, C_{\ell}(0,0) - C_{\ell}(\lambda\vec{f},0) \; - S(\lambda\vec{f},\vec{\vartheta})] \; . \qquad (4.51)$$

It is insightful to separate this result into three parts. These are: 1) a diffraction limited term, $\mathcal{I}_{DL}(\vec{f})$, 2) a random apodization term, $\exp[\, C_{\ell}(0,0) - C_{\ell}(\lambda\vec{f},0)]$, which for most cases of interest can be approximated by $\exp[\, C_{\ell}(0,0)\,]$, and 3) the anisoplanatism term, $\exp[\,-S(\lambda\vec{f},\vec{\vartheta})\,]$.

For the evaluation of the performance of an adaptive optics imaging system with compensation of random apodization we use Eq.'s (4), (5), (6), (18), (32), and (34). When we twice substitute Eq. (6) into Eq. (5) and simplify we get

$$\langle B_{IW} \rangle = \{ \int d\vec{r} \, \tfrac{1}{2} A^* A \, W(\vec{r}) \, E_{IW}(0) \}^{-1}$$

$$= \{ \tfrac{1}{2} A^* A \, (\tfrac{1}{4} \pi D^2) \, E_{IW}(0) \}^{-1} \quad . \tag{4.52}$$

From consideration of Eq.'s (32) and (34) it is obvious that

$$E_{IW}(0) = \exp \{ 4 [C_\ell(0,0) - C_\ell(0, \vec{\vartheta})] \} \quad , \tag{4.53}$$

so that

$$\langle B_{IW} \rangle = \tfrac{1}{2} A^* A \, (\tfrac{1}{4} \pi D^2) \exp \{ 4 [C_\ell(0,0) - C_\ell(0,\vec{\vartheta})] \}^{-1} \, . \tag{4.54}$$

When Eq. (6) is twice substituted into Eq. (4), after suitable rearrangement of terms we get

$$\langle \mathcal{I}_{IW}(\vec{f}) \rangle = \tfrac{1}{2} \langle B_{IW} \rangle \int d\vec{r} \; A^* A \; W(\vec{r} + \lambda \vec{f}) \, W(\vec{r}) \; E_{IW}(\lambda \vec{f})$$

$$= \tfrac{1}{2} A^* A \, \langle B_{IW} \rangle \, E_{IW}(\lambda \vec{f}) \int d\vec{r} \; W(\vec{r} + \lambda \vec{f}) \, W(\vec{r})$$

$$= \tfrac{1}{2} A^* A \, \langle B_{IW} \rangle \, E_{IW}(\lambda \vec{f}) \, (\tfrac{1}{4} \pi D^2) \; K(\lambda \vec{f}) \quad , \tag{4.55}$$

where in writing this equation we have suppressed a factor of $\exp(ik\lambda\vec{f} \cdot \vec{\theta}_T)$ as representing nothing more than the phase shift to be associated with the fact that the object we are imaging is displaced an angular distance $\vec{\theta}_T$ from the z-axis. Now making use of Eq. (2.22), (54), (44), (32), and (34) we can write this as

$$\langle \mathcal{I}_{IW}(\vec{f}) \rangle = \mathcal{I}_{DL}(\vec{f}) \exp [-S(\lambda \vec{f}, \vec{\vartheta})] \quad . \tag{4.56}$$

In this case there is simply the diffraction limited contribution, $\mathcal{I}_{DL}(\vec{f})$, and the anisoplanatism contribution, $\exp[-S(\lambda\vec{f}, \vec{\vartheta})]$.

To evaluate the performance of an adaptive optics laser transmitter without random apodization compensation we shall make use of Eq.'s (7), (8), (9), (10), (11), (19), (35), and (36). When we substitute Eq. (11) twice into Eq. (10) and suitably simplify we obtain the result that

$$\langle P_{LTW/O} \rangle = \tfrac{1}{2} A^* A \int d\vec{r} \; W(\vec{r})$$

$$= \tfrac{1}{2} A^* A \, (\tfrac{1}{4} \pi D^2) \quad . \tag{4.57}$$

When we substitute Eq. (9) twice into Eq. (8), make a double integral of the product of integrals, interchange the order of integration and ensemble averaging, and otherwise suitably manipulate terms, we find that the result can be written as

$$\langle \mathcal{P}_{LTW/O} \rangle = \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \iint d\vec{r}' d\vec{r} \; W(\vec{r}') \; W(\vec{r}) \; E_{LTW/O} \, (\vec{r}' - \vec{r})$$

$$= \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \iint d\vec{\rho} \; d\vec{r} \; W(\vec{r} + \vec{\rho}) \; W(\vec{r}) \; E_{LTW/O}(\vec{\rho})$$

$$= \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \left(\tfrac{1}{4} \pi D^2\right) \int d\vec{\rho} \; K(\vec{\rho}) \; E_{LTW/O}(\vec{\rho}) \; . \qquad (4.58)$$

From consideration of Eq.'s (35) and (36) we see that this can be rewritten as

$$\langle \mathcal{P}_{LTW/O} \rangle = \tfrac{1}{2} A^* A \lambda^{-2} R^{-2} \left(\tfrac{1}{4} \pi D^2\right) \int d\vec{\rho} \, K(\vec{\rho}) \exp\left[-C_\ell(0,0) + C_\ell(\vec{\rho}, 0) - S(\vec{\rho}, \vec{\vartheta})\right] . \qquad (4.59)$$

On substituting Eq.'s (57) and (59) into Eq. (7), and making use of Eq.'s (2.85) and (2.32) we get the result that

$$\langle G_{LTW/O} \rangle = \lambda^{-2} \int d\vec{\rho} \, K(\rho) \exp\left[C_\ell(0,0) - C_\ell(\vec{\rho}, 0) - S(\vec{\rho}, \vec{\vartheta})\right]$$

$$= \tfrac{1}{4} \pi (D/\lambda)^2 \; \frac{\int d\vec{\rho} \; K(\rho) \exp\left[C_\ell(0,0) - C_\ell(\vec{\rho}, 0) - S(\vec{\rho}, \vec{\vartheta})\right]}{\int d\vec{\rho} \; K(\rho)}$$

$$= G_{DL} \left\{ \frac{\int d\vec{\rho} \; K(\rho) \exp\left[C_\ell(0,0) - C_\ell(\vec{\rho}, 0) - S(\vec{\rho}, \vec{\vartheta})\right]}{\int d\vec{\rho} \; K(\rho)} \right\} . \qquad (4.60)$$

It is convenient at this point to take account of the fact that for almost all cases of interest the aperture diameter of the laser transmitter is very much larger than the range of the log-amplitude covariance function, $C_\ell(\vec{\rho}, 0)$. This means that over most of the range of integration, $C_\ell(\vec{\rho}, 0)$ is very nearly equal to zero. This allows us to rewrite Eq. (60) in the approximate form

$$\langle G_{LTW/O} \rangle = G_{DL} \exp\left[C_{\ell}(0,0)\right]\left\{\frac{\int d\vec{\rho}\, K(\rho) \exp\left[-S(\vec{\rho},\vartheta)\right]}{\int d\vec{\rho}\, K(\rho)}\right\}. \qquad (4.61)$$

We can identify the three parts of this result with 1) diffraction limits, 2) random apodization effects, and 3) anisoplanatism effects.

Finally, to evaluate the performance of an adaptive optics laser transmitter with compensation for random apodization we shall make use of Eq.'s (12), (13), (14), (15), (16), (20), (21), (39), (42), and (43). We start by noting that if we substitute Eq. (16) into Eq. (15) twice, suitably manipulate the terms, and take note of Eq. (21) we get the result that

$$\langle P_{LTW} \rangle = \tfrac{1}{2} A^{*} A \int d\vec{r}\; W(\vec{r})\; E_{LTW2} \qquad . \qquad (4.62)$$

Making use of Eq. (43) this reduces to

$$\begin{aligned} \langle P_{LTW} \rangle &= \tfrac{1}{2} A^{*} A \int d\vec{r}\; W(\vec{r}) \\ &= \tfrac{1}{2} A^{*} A \;(\tfrac{1}{4}\pi D^2) \qquad . \qquad (4.63) \end{aligned}$$

When we substitute Eq. (14) twice into Eq. (13), make a double integral of the product of integrals, and interchange the order of integration and ensemble averaging, then on suitable simplification and making reference to Eq. (20) we see that the result can be written as

$$\begin{aligned} \langle \mathcal{P}_{LTW} \rangle &= \tfrac{1}{2} A^{*} A\, \lambda^{-2} R^{-2} \iint d\vec{r}\,'\, d\vec{r}\; W(\vec{r}\,')\, W(\vec{r})\;\; E_{LTW}(\vec{r}\,' - \vec{r}) \\ &= \tfrac{1}{2} A^{*} A\, \lambda^{-2} R^{-2} \iint d\vec{\rho}\, d\vec{r}\;\; W(\vec{r}+\vec{\rho})\;\, W(\vec{r})\;\, E_{LTW}(\vec{\rho}) \\ &= \tfrac{1}{2} A^{*} A\, \lambda^{-2} R^{-2}\, (\tfrac{1}{4}\pi D^2) \int d\vec{\rho}\;\, K(\rho)\;\, E_{LTW}(\vec{\rho})\;. \qquad (4.64) \end{aligned}$$

From consideration of Eq.'s (39) and (42) we see that this can be rewritten as

$$\langle \mathcal{P}_{LTW} \rangle = \tfrac{1}{2} A^{*} A\, \lambda^{-2} R^{-2} (\tfrac{1}{4}\pi D^{2}) \int d\vec{\rho}\; K(\vec{\rho}) \exp\left[-S(\vec{\rho},\vec{\vartheta})\right]. \qquad (4.65)$$

On substituting Eq. (63) and (65) into Eq. (12), and (as before) making use of Eq.'s (2.85) and (2.32) we get the results that

$$\langle G_{LTM} \rangle = \lambda^{-2} \int d\vec{\rho}\, K(\rho) \exp[-S(\vec{\rho},\vec{\vartheta})]$$

$$= \tfrac{1}{4}\pi(D/\lambda)^2 \frac{\int d\vec{\rho}\, K(\vec{\rho}) \exp[-S(\vec{\rho},\vec{\vartheta})]}{\int d\vec{\rho}\, K(\vec{\rho})}$$

$$= G_{DL} \left\{ \frac{\int d\vec{\rho}\, K(\vec{\rho}) \exp[-S(\vec{\rho},\vec{\vartheta})]}{\int d\vec{\rho}\, K(\rho)} \right\} \qquad . \qquad (4.66)$$

The interpretation of this result is obvious in the sense that the antenna gain is determined by two parts, a diffraction limited part and an anisoplanatism part.

1.4.3 Performance Resume

An examination of the results of the preceding subsection makes it clear, as we might have expected, that aside from a performance factor reasonably well approximated by

$$\mathfrak{D}_{RA} = \exp[C_{\ell}(0,0)] \qquad , \qquad (4.67)$$

there is no significant difference between the performance of adaptive optics systems designed to correct for random apodization and those not so designed. This random apodization factor, $\mathfrak{D}_{RA}$, is generally quite close to unity*, so in general we need not be concerned about random apodization.

Ignoring the random apodization we can write for the ensemble average OTF of an adaptive optics imaging system

$$\langle \mathfrak{T}(\vec{f}) \rangle = \mathfrak{T}_{DL} \exp[-S(\lambda\vec{f}, \vec{\vartheta})] \qquad . \qquad (4.68)$$

Similarly ignoring random apodization considerations we can write for the ensemble average antenna gain of an adaptive optics laser transmitter

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* Implicit in this is the assumption that our propagation path is such that the log-amplitude variance, $C_{\ell}(0,0)$, is less than one-half, i.e., that scintillation saturation has not occurred.

$$\langle G \rangle = G_{DL} \left\{ \frac{\int d\vec{\rho}\ K(\rho) \exp[-S(\vec{\rho}, \vec{\vartheta})]}{\int d\vec{\rho}\ K(\rho)} \right\} \quad . \tag{4.69}$$

We recall here that we are dealing with an aperture of diameter D and a wavelength λ . We further recall that

$$K(\vec{\rho}) = \begin{cases} \frac{2}{\pi} \{ \cos^{-1}(\rho/D) - (\rho/D)[1 - (\rho/D)^2]^{1/2} \}, & \text{if } \rho \le D \\ 0 , & \text{if } \rho > D \end{cases} \quad , \tag{4.70}$$

that

$$\mathfrak{T}_{DL}(\vec{f}) = K(\lambda f) \quad , \tag{4.71}$$

and

$$G_{DL} = \tfrac{1}{4}\pi (D/\lambda)^2 \quad . \tag{4.72}$$

In addition we note that

$$S(\vec{\rho}, \vec{\vartheta}) = 2.905\, k^2 \int_0^R dv\ C_N^2 \left(\rho_v^{5/3} + \vartheta_v^{5/3} - \tfrac{1}{2} |\vec{\rho}_v + \vec{\vartheta}_v|^{5/3} - \tfrac{1}{2} |\vec{\rho}_v - \vec{\vartheta}_v|^{5/3} \right), \tag{4.73}$$

where

$$\vec{\rho}_v = \vec{\rho}\,(R - v)/R \quad , \tag{4.74}$$

and

$$\vec{\vartheta}_v = \vec{\vartheta}\, v \quad . \tag{4.75}$$

These last nine equations fairly well sum up the results of all of the preceding analysis. At this point the remaining task is to understand the nature of the $S(\vec{\rho}, \vec{\vartheta})$ - function (which we call the basic anisoplanatism function) and its implications in Eq.'s (68) and (69). We take this up in the next section.

1.5 Numerical Results

In this section we shall develop numerical results from the analytic expressions presented in the previous section. We shall be interested in developing an understanding of the nature of the dependence of anisoplanatism effects on the various problem parameters, as evidenced by the numerical results. We shall first undertake an examination of the quantity $S(\vec{\rho}, \vec{\vartheta})$, which is directly related to the anisoplanatism effects in adaptive optics imaging. With those results in hand we will then take up consideration of anisoplanatism effect on antenna gain, $\langle G \rangle$, of an adaptive optics laser transmitter. These two matters are treated in the following subsections.

1.5.1 Adaptive Optics Imaging

According to Eq. (4.68) the OTF of an adaptive optics imaging system as influenced by anisoplanatism effects can be written as

$$\langle \mathfrak{T}(\vec{f}) \rangle = \mathfrak{T}_{DL}(\vec{f}) \exp[-S(\lambda\vec{f}, \vec{\vartheta})] \quad , \qquad (5.1)$$

for a spatial frequency of $\vec{f}$, an angular separation between the direction to the beacon and to the object to be imaged of $\vec{\vartheta}$, and an operating wavelength of λ. Here $\mathfrak{T}_{DL}(\vec{f})$ is the defraction limited OTF [defined by Eq. (4.71)], and the basic anisoplanatism function, $S(\lambda\vec{f}, \vec{\vartheta})$, is defined in Eq. (4.73). Replacing the length $\lambda\vec{f}$ by the length $\vec{r}$ we can write

$$S(\vec{r}, \vec{\vartheta}) = 2.905\, k^2 \int_0^R dv\; C_N^2 \{ r^{5/3}[1-(v/R)]^{5/3} + \vartheta^{5/3} v^{5/3} - \tfrac{1}{2} |\vec{r}[1-(v/R)] + \vec{\vartheta} v|^{5/3} - \tfrac{1}{2} |\vec{r}[1-(v/R)] - \vec{\vartheta} v|^{5/3} \} . \qquad (5.2)$$

If we let c denote the cosine of the angle between the orientation of the two vectors, $\vec{r}$ and $\vec{\vartheta}$, so that

$$c = \vec{r} \cdot \vec{\vartheta} / (|\vec{r}|\, |\vec{\vartheta}|) \qquad (5.3)$$

then we can rewrite Eq. (2) as

$$S(\vec{r},\vec{\vartheta}) = 2.905\, k^2 \int_0^R dv\; C_N^2 \,\{ r^{5/3}[1-(v/R)]^{5/3} + \vartheta^{5/3}\, v^{5/3}$$

$$-\tfrac{1}{2}\{r^2[1-(v/R)]^2 + 2\, r\,\vartheta\,[1-(v/R)]\, v\, c + \vartheta^2\, v^2\}^{5/6}$$

$$-\tfrac{1}{2}\{r^2[1-(v/R)]^2 - 2\, r\,\vartheta\,[1-(v/R)]\, v\, c + \vartheta^2\, v^2\}^{5/6}\,\} . \qquad (5.4)$$

It is convenient at this point to consider the form this expression takes for the two limiting cases of r very large and of ϑ very large.

In the limiting case of r very large compared to any value of ϑv (for values of v within the range of integration and for which C_N^2 is not negligibly small) we can write

$$\lim_{\vec{r}\to\infty} S(\vec{r},\vec{\vartheta}) = S_r(\vec{r},\vec{\vartheta}) \quad , \qquad (5.5)$$

where

$$S_r(\vec{r},\vec{\vartheta}) \approx 2.905\, k^2 \int_0^R dv\; C_N^2\, \vartheta^{5/3}\, v^{5/3} \quad . \qquad (5.6)$$

The form of the integrand in Eq. (6) follows from Eq. (4) when we consider that for r very large the two curly bracket terms in Eq. (4) each reduce to approximately $r^{5/3}[1-(v/R)]^{5/3}$, and so together approximately cancel the term of this form already present in the integrand. Thus only the $\vartheta^{5/3} v^{5/3}$-term is left in the integrand — resulting in Eq. (6). If we define the isoplanatic patch angle, ϑ_0, by the equation

$$\vartheta_0 = \{ 2.905\, k^2 \int_0^R dv\; C_N^2\, v^{5/3} \}^{-3/5} \quad . \qquad (5.7)$$

then we can rewrite Eq. (6) in very simple form as

$$S_r(\vec{r},\vec{\vartheta}) \approx (\vartheta/\vartheta_0)^{5/3} \quad . \tag{5.8}$$

For the alternate extreme limiting case of ϑ very large compared to almost all values of r/v we can write

$$\lim_{\vartheta\to\infty} S(\vec{r},\vec{\vartheta}) = S_\theta(\vec{r},\vec{\vartheta}) \quad , \tag{5.9}$$

where

$$S_\theta(\vec{r},\vec{\vartheta}) \approx 2.905\, k^2 \int_0^R dv\ C_N^2\ r^{5/3}\ [1-(v/R)]^{5/3} \quad . \tag{5.10}$$

The form of the integrand in Eq.(10) follows from Eq. (4) when we consider that for ϑ very large the two curly brackets terms in Eq. (4) each reduce, almost everywhere, to approximately $\vartheta^{5/3}\ v^{5/3}$, and so together approximately cancel the term of this form already present in the integrand. Thus only the $r^{5/3}\ [1-(v/R)]^{5/3}$-term is left in the integrand — resulting in Eq. (10). When we recall that the value of the effective coherence diameter, r_0, is given by the expression[11]

$$r_0 = \{(2.905/6.88)\, k^2 \int_0^R dv\ C_N^2\ [1-(v/R)]^{5/3}\}^{-3/5} \quad , \tag{5.11}$$

then it is easy to see that we can rewrite Eq. (10) in the rather simple form

$$S_\theta(\vec{r},\vec{\vartheta}) \approx 6.88\, (r/r_0)^{5/3} \quad . \tag{5.12}$$

In conjunction with these two characteristic propagation parameters, r_0 and ϑ_0, we introduce the "effective path length" parameter, $\mathcal{L}_0$, which we define by the equation

$$\mathcal{L}_0 = r_0/\vartheta_0 \tag{5.13}$$

With these three parameters in hand we are now ready to numerically examine the behavior of $S(\vec{r}, \vec{\vartheta})$.

The basic anisoplanatism function, $S(\vec{r}, \vec{\vartheta})$ is a function of two parameters and as such is awkward for numerical evaluation and for graphical examination. However, as can be seen from a study of Eq. (4), if we extract a factor of $r^{5/6}\ \vartheta^{5/6}$ from inside the integrand, the resulting integral appears to depend only on r/ϑ and not on r or ϑ separately, i. e., it is a one parameter function. This suggests that we write in place of Eq. (4)

$$S(\vec{r}, \vec{\vartheta}) = 2.905\ k^2\ (r\ \vartheta)^{5/6} \int_0^R dv\ C_N{}^2 \Big\langle\!\!\Big\langle (r/\vartheta)^{5/6}[1-(v/R)]^{5/3} + (\vartheta/r)^{5/6}\ v^{5/3}$$

$$-\tfrac{1}{2}\{(r/\vartheta)[1-(v/R)]^2 + 2[1-(v/R)]\ v\ c + (\vartheta/r)\ v^2\}^{5/6}$$

$$-\tfrac{1}{2}\{(r/\vartheta)[1-(v/R)]^2 - 2[1-(v/R)]\ v\ c + (\vartheta/r)\ v^2\}^{5/6}\Big\rangle\!\!\Big\rangle. \quad (5.14)$$

It is convenient to make use of the propagation parameters, r_0, ϑ_0, and $\mathcal{L}_0$, to normalize this expression and accordingly we write

$$S(\vec{r}, \vec{\vartheta}) = 2.905\ k^2\ (r_0\ \vartheta_0\ \mathcal{L}_0)^{5/6}[(r/r_0)(\vartheta/\vartheta_0)]^{5/6} \int_0^R dv\ C_N{}^2$$

$$\times \Big\langle\!\!\Big\langle (\vartheta\mathcal{L}_0/r)^{-5/6}[1-(v/R)]^{5/3} + (\vartheta\mathcal{L}_0/r)^{5/6}\ v^{5/3}\ \mathcal{L}_0^{-5/3}$$

$$-\tfrac{1}{2}\{(\vartheta\mathcal{L}_0/r)^{-1}[1-(v/R)]^2 + 2[1-(v/r)]\ v\ \mathcal{L}_0^{-1}\ c + (\vartheta\mathcal{L}_0/r)\ v^2\mathcal{L}_0^{-2}\}^{5/6}$$

$$-\tfrac{1}{2}\{(\vartheta\mathcal{L}_0/r)^{-1}[1-(v/R)]^2 - 2[1-(v/r)]\ v\mathcal{L}_0^{-1}\ c + (\vartheta\mathcal{L}_0/r_0)\ v^2\mathcal{L}_0^{-2}\}^{5/6} \quad (5.15)$$

Taking note of Eq. (13) this can be rewritten as

$$S(\vec{r}, \vec{\vartheta}) = [(r/r_0)(\vartheta/\vartheta_0)]^{5/6}\ \mathcal{J}[(r/r_0)/(\vartheta/\vartheta_0)] \quad (5.16)$$

where

$$\mathscr{J}(Q) = 2.905\, k^2 (r_0\, \vartheta_0\, \mathcal{L}_0)^{5/6} \int_0^R dv\ C_N^2 \Big(Q^{5/6}[1-(v/R)]^2 + Q^{-5/6} v^{5/3} \mathcal{L}_0^{-5/3}$$

$$- \tfrac{1}{2}\{Q[1-(v/R)]^2 + 2[1-(v/R)]\, v\, \mathcal{L}_0^{-1} c + Q^{-1} v^2 \mathcal{L}_0^{-2}\}^{5/6}$$

$$- \tfrac{1}{2}\{Q[1-(v/R)]^2 - 2[1-(v/R)]\, v\, \mathcal{L}_0^{-1} c + Q^{-1} v^2 \mathcal{L}_0^{-2}\}^{5/6} \Big) . \quad (5.17)$$

Further, taking note of Eq. (13) along with Eq. (11), this can be reduced to the form

$$\mathscr{J}(Q) = 6.88 \left\{ \int_0^R dv\ C_N^2\ [1-(v/R)]^{5/3} \right\}^{-1} \int_0^R dv\ C_N^2$$

$$\times \Big(Q^{5/6}\, [1-(v/R)]^{5/3} + Q^{-5/6}\, v^{5/3}\, \mathcal{L}_0^{-5/3}$$

$$- \tfrac{1}{2}\{Q[1-(v/R)]^2 + 2[1-(v/R)]\, v\, \mathcal{L}_0^{-1}\, c + Q^{-1}\, v^2\, \mathcal{L}_0^{-2}\}^{5/6}$$

$$- \tfrac{1}{2}\{Q[1-(v/R)]^2 - 2[1-(v/R)]\, v\, \mathcal{L}_0^{-1}\, c + Q^{-1}\, v^2\, \mathcal{L}_0^{-2}\}^{5/6} \Big) . \quad (5.18)$$

To see how this function, $\mathscr{J}(Q)$, behaves we have developed numerical results for the two cases of 1) a uniform propagation path, i. e. , one in which the strength of turbulence, and thus the value of C_N^2, does not vary with position along the propagation path, and 2) a ground-to-space path for which the strength of turbulence, and thus C_N^2, varies with altitude in the manner shown in Fig. 1.

For the ground-to-space path it can be shown that the value of $\mathscr{J}(Q)$ is independent of the zenith angle, Ψ. [Of course for given values of r and ϑ, since r_0 and ϑ_0 depend on the zenith angle, the appropraite value of Q will depend on Ψ, but $\mathscr{J}(Q)$ considered as a function of Q <u>per se</u>, does not depend on Ψ.] To see why $\mathscr{J}(Q)$ is independent of Ψ we first note that according to Eq.'s (7) and (11) ϑ_0 is proportional to $[\sec(\Psi)]^{-8/5}$ while r_0 is proportional to $[\sec(\Psi)]^{-3/5}$. Thus it follows from Eq. (13) that $\mathcal{L}_0$ is proportional to $\sec(\Psi)$. Examining Eq. (18) we note first that all the (v/R)-terms can be

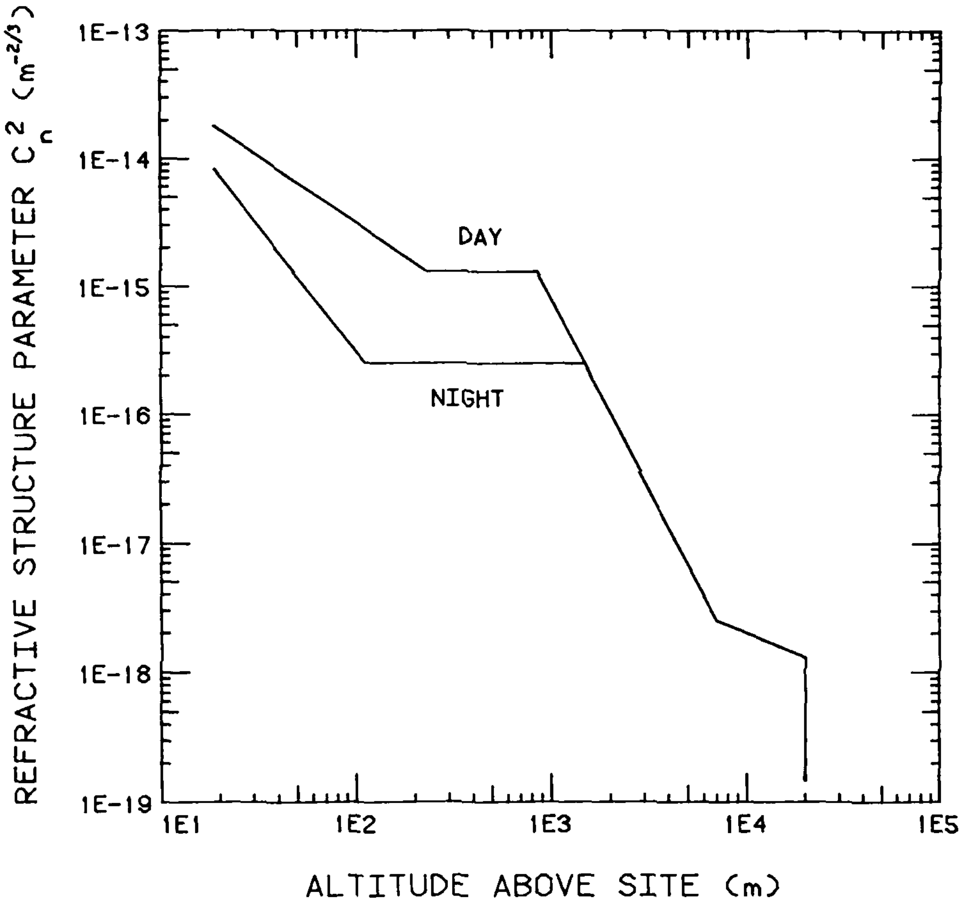

Figure 1. Verona Turbulence Model

dropped since for ground-to-space we can consider R to be essentially infinite, and second that all other v-dependencies are actually $(v/\mathcal{L}_0)$-dependencies. Thus it follows that in the two integrals in Eq. (18) [except for the two dv-factors, each proportional to sec (Ψ) when expressed in altitude, h, rather than path length, v, dependent form, and thus having no composite sec (Ψ)-dependence] the integrand has no sec (Ψ)-dependent terms. This means that we can calculate $\mathcal{J}(Q)$ for the vertical ($\Psi = 0$) ground-to-space case and consider the results to apply to the case of ground-to-space propagation at any zenith angle.

In Fig's 2, 3, and 3' we show for the uniform path and the ground-to-space path, the form of $\mathcal{J}(Q)$ as a function of Q for the orientation angles between $\vec{r}$ and $\vec{\vartheta}$ such that c takes the values 0.00, 0.25, 0.50, 0.75, and 1.00. In studying these results we note that in the case of small enough values of Q, $\mathcal{J}(Q)$ behaves approximately as 6.88 $Q^{5/6}$,

$$\lim_{Q \to 0} \mathcal{J}(Q) \approx 6.88 \; Q^{5/6} \quad , \qquad (5.19)$$

while for large enough values of Q, $\mathcal{J}(Q)$ behaves approximately as $Q^{-5/6}$,

$$\lim_{Q \to \infty} \mathcal{J}(Q) \approx Q^{-5/6} \quad . \qquad (5.20)$$

These limiting behavior results are in agreement with the limiting results for S $(\vec{r}, \vec{\vartheta})$ presented in Eq.'s (8) and (12).

Before passing from our consideration of the behavior of S $(\vec{r}, \vec{\vartheta})$ and of $\mathcal{J}(Q)$ to take up consideration of the adaptive optics laser transmitters antenna gain we wish to first look at approximations for $\mathcal{J}(Q)$ [and thus for S $(\vec{r}, \vec{\vartheta})$] which will be accurate a bit closer to the $Q = 1$ region than Eq.'s (19) and (20). To obtain such results we start by noting that by retaining the

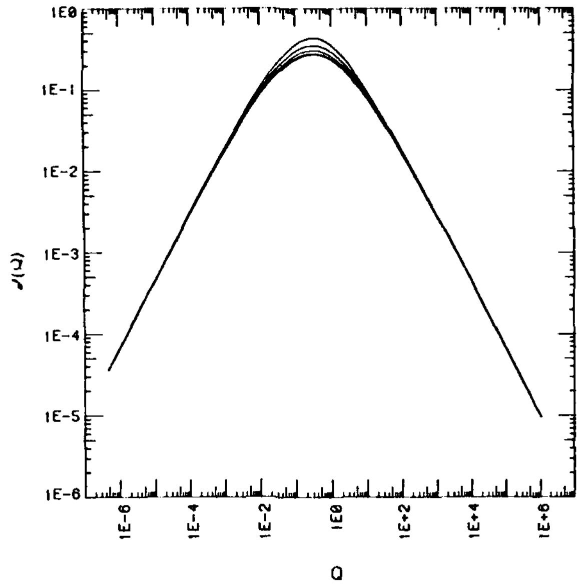

Figure 2. Anisoplanatism Function, $\mathcal{J}(Q)$, For a Horizontal Path.

Results are shown for $c = 0.00$, 0.25, 0.50, 0.75, and 1.00 with $c = 1.00$ corresponding to the highest curve and $c = 0.00$ to the lowest. The $c = 0.25$ curve is so near coincident with the $c = 0.00$ curve as to be hardly distinguishable.

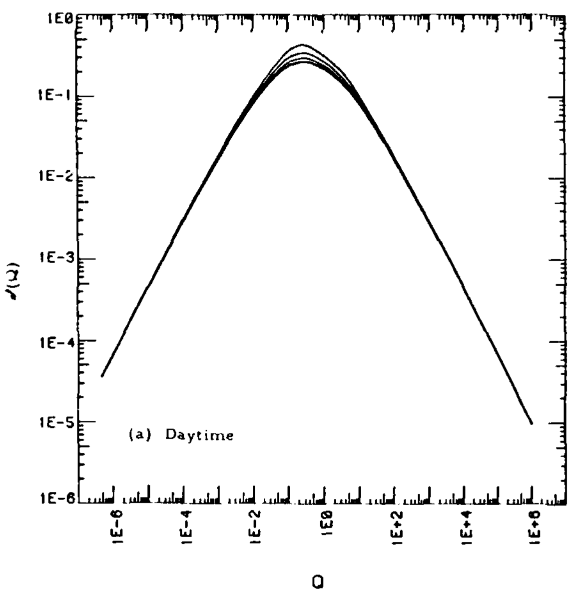

Figure 3. Anisoplanatism Function, $\mathcal{J}(Q)$, For a Daytime Vertical Path.

Results are shown for $c = 0.00$, 0.25, 0.50, 0.75, and 1.00 with $c = 1.00$ corresponding to the highest curve and $c = 0.00$ to the lowest. The $c = 0.25$ curve is so near coincident with the $c = 0.00$ curve as to be hardly distinguishable.

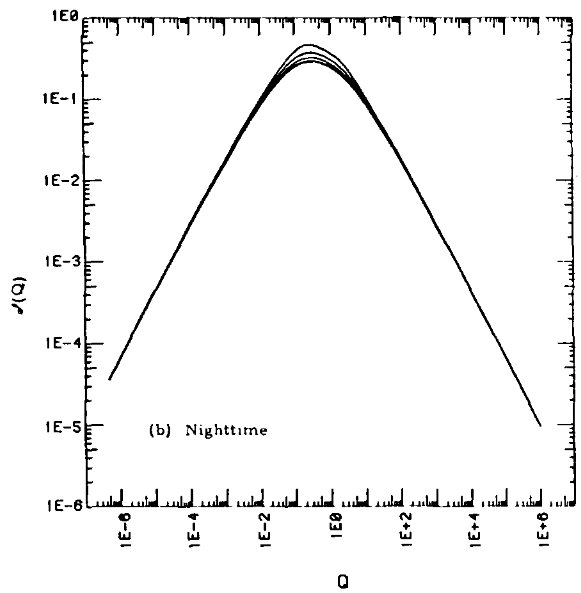

Figure 3'. Anisoplanatism Function, $\mathcal{J}(Q)$, For a Nighttime Vertical Path.

Results are shown for $c = 0.00$, 0.25, 0.50, 0.75, and 1.00, with $c = 1.00$ corresponding to the highest curve and $c = 0.00$ to the lowest. The $c = 0.25$ curve is so near coincident with the $c = 0.00$ curve as to be hardly distinguishable.

leading terms in a power series expansion we can write

$$F(Q) \approx Q^{5/6}\,[1-(v/R)]^{5/3}\{1+\tfrac{5}{6}(1-\tfrac{1}{3}c^2)\,Q^{-2}[1-(v/R)]^{-2}\,v^2\,\mathcal{L}_0^{-2}\}\,,$$

$$\text{if } Q \gg 1\,, \qquad (5.21)$$

and

$$F(Q) \approx Q^{-5/6}\,v^{5/3}\,\mathcal{L}_0^{-5/3}\{1+\tfrac{5}{6}(1-\tfrac{1}{3}c^2)\,Q^2\,[1-(v/R)]^2\,v^{-2}\,\mathcal{L}_0^2\}\,,$$

$$\text{if } Q \ll 1\,, \qquad (5.22)$$

where for convenience we have used F (Q) defined as follows;

$$F(Q) = \tfrac{1}{2}\{Q[1-(v/R)]^2+2\,[1-(v/R)]\,v\,\mathcal{L}_0^{-1}\,c+Q^{-1}\,v^2\,\mathcal{L}_0^{-2}\}^{5/6}$$

$$+\tfrac{1}{2}\{Q[1-(v/R)^2-2\,[1-(v/R)]\,v\,\mathcal{L}_0^{-1}\,c+Q^{-1}v^2\,\mathcal{L}_0^{-2}\}^{5/6}. \quad (5.23)$$

Making use of Eq. (21) we can obtain from Eq. (18) the result that

$$\mathcal{J}(Q) \approx 6.88\,\{\int_0^R dv\;C_N^2\;[1-(v/R)]^{5/3}\}^{-1}\int_0^R dv\;C_N^2$$

$$\times\{Q^{-5/6}\,v^{5/3}\,\mathcal{L}_0^{-5/3}-\tfrac{5}{6}(1-\tfrac{1}{3}c^2)\,Q^{-7/6}\,[1-(v/R)]^{-1/3}\,v^2\,\mathcal{L}_0^{-2}\},$$

$$\text{if } Q \gg 1\,. \qquad (5.24)$$

Similarly, from Eq.'s (22) and (18) we obtain the result that

$$\mathcal{J}(Q) \approx 6.88\,\{\int_0^R dv\;C_N^2\;[1-(v/R)]^{5/3}\}^{-1}\int_0^R dv\;C_N^2$$

$$\times\{Q^{5/6}[1-(v/R)]^{5/3}-\tfrac{5}{6}(1-\tfrac{1}{3}c^2)\,Q^{7/6}\,[1-(v/R)]^2\,v^{-1/3}\,\mathcal{L}_0^{1/3}\}\,,$$

$$\text{if } Q \ll 1\,. \qquad (5.25)$$

It is convenient to rewrite these results as

$$\mathcal{J}(Q) \approx A \; Q^{-5/6} \; [\; 1 - \alpha(1 - \tfrac{1}{3} c^2) \; Q^{-1/3} \;] \,, \quad \text{if} \quad Q \gg 1 \,, \tag{5.26}$$

and

$$\mathcal{J}(Q) \approx B \; Q^{5/6} \; [\; 1 - \beta \,(\, 1 - \tfrac{1}{3} c^2) \; Q^{1/3} \;] \,, \text{ if } \quad Q \ll 1 \,, \tag{5.27}$$

where

$$A = 6.88 \, \mathcal{L}_0^{-5/3} \, \{ \int_0^R dv \;\; C_N^2 \; v^{5/3} \; \} \, \{ \int_0^R dv \;\; C_N^2 \; [1 - (v/R)]^{5/3} \; \}^{-1} , \tag{5.28}$$

$$\alpha = \tfrac{5}{6} \, \mathcal{L}_0^{-1/3} \, \{ \int_0^R dv \;\; C_N^2 \; [1 - (v/R)]^{-1/3} \; v^2 \; \} \, \{ \int_0^R dv \;\; C_N^2 \; v^{5/3} \; \}^{-1} , \tag{5.29}$$

$$B = 6.88 \qquad , \tag{5.30}$$

$$\beta = \tfrac{5}{6} \, \mathcal{L}_0^{1/3} \, \{ \int_0^R dv \;\; C_N^2 \; [1 - (v/R)]^2 \; v^{-1/3} \; \} \, \{ \int_0^R dv \;\; C_N^2 \; [1 - (v/R)]^{5/3} \}^{-1} . \tag{5.31}$$

Making use of Eq.'s (7), (11), and (13) we can rewrite Eq. (28) as

$$A = 1 \qquad , \tag{5.32}$$

as we might have expected from consideration of Eq. (20).

To see just how well Eq.'s (26) and (27) approximate $\mathcal{J}(Q)$ in the vicinity of $Q = 1$, in Fig.'s 4, 5, and 5′ we have replotted the curves of Fig.'s 2, 3, and 3′ for $c = 0.0$, and 1.0. These results are shown as the solid lines. Superimposed on these we have plotted $\mathcal{J}(Q)$ as calculated from Eq. (26) for Q greater than or equal to unity and from Eq. (27) for Q less than or equal to unity. As can be seen these approximations are quite good

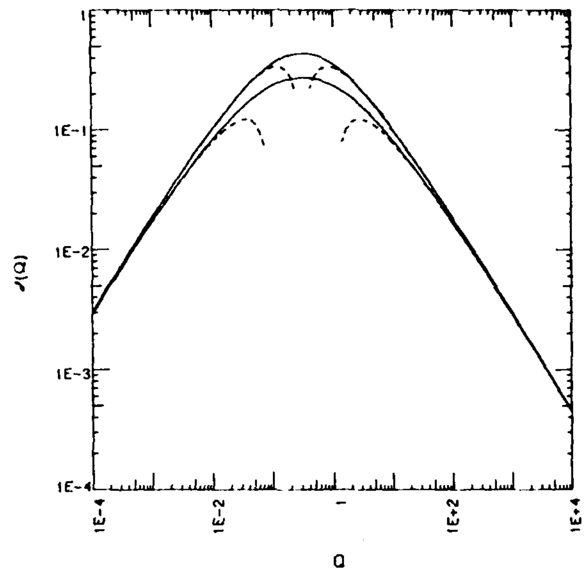

Figure 4. Asymptotic Approximation to $\mathcal{J}(Q)$ for a Horizontal Path.

The solid lines correspond to the $c = 0.00$ and $c = 1.00$ curves from Fig. 2, while the dashed lines show the asymptotic approximation.

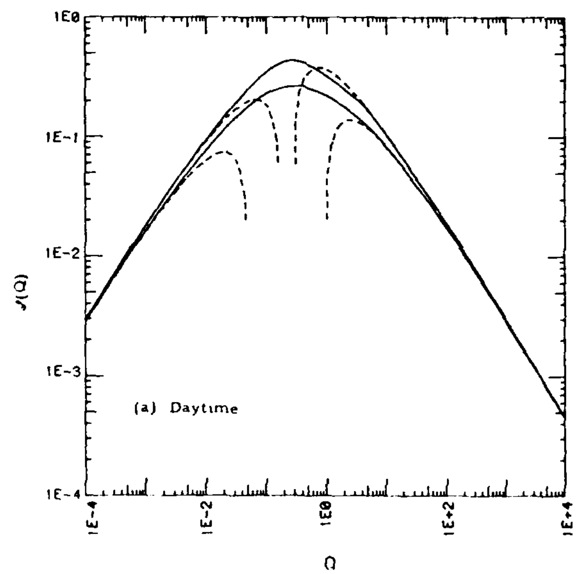

Figure 5. Asymptotic Approximation to $\mathcal{J}(Q)$ for a Daytime Vertical Path.

The solid lines correspond to the $c = 0.00$ and $c = 1.00$ curves from Fig. 3, while the dashed lines show the asymptotic approximation.

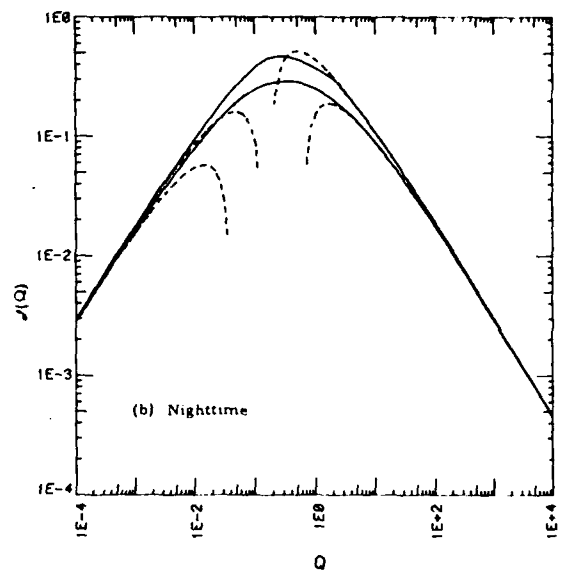

Figure 5′. Asymptotic Approximation to $\mathcal{J}(Q)$ for a Nighttime Vertical Path.

The solid lines correspond to the $c = 0.00$ and $c = 1.00$ curves from Fig. 3′, while the dashed lines show the asymptotic approximations.

outside the range from $Q = 0.05$ to $Q = 2.0$.

1.5.2 Adaptive Optics Laser Transmitter

With these results in hand, particularly the exact evaluations of $\mathscr{J}(Q)$ presented in Fig.'s 2, 3, and 3' we are now ready to compute adaptive optics system performance. The imaging system OTF results are so directly related to the $\mathscr{J}(Q)$ curves that there is no need for us to examine these results in detail. Accordingly we turn our attention to the laser transmitter antenna gain results. The basic result, for which we wish to obtain results is for the normalized antenna gain, $\langle G \rangle / G_{DL}$, which we can write in accordance with Eq. (4.69), as

$$\langle G \rangle / G_{DL} = \left(\tfrac{1}{4} \pi D^2 \right)^{-1} \int d\vec{r} \; K(r) \; \exp\left[-S(\vec{r}, \vec{\vartheta}) \right], \tag{5.33}$$

where

$$K(r) = \begin{cases} \frac{2}{\pi} \left\{ \cos^{-1}(r/D) - (r/D)\left[1 - (r/D)^2 \right]^{1/2} \right\}, & \text{if } r \le D \\ 0, & \text{if } r > D \end{cases}, \tag{5.34}$$

for a circular clear aperture of diameter D. In obtaining Eq. (33) from Eq. (4.69) we have made use of the fact that

$$\int d\vec{r} \; K(r) = \tfrac{1}{4} \pi D^2 . \tag{5.35}$$

In Fig. 6 we show results for the normalized antenna gain as a function of the normalized aperture diameter, D/r_0 for a horizontal (C_N^2 constant) path. Results are shown for normalized angular separations, ϑ/ϑ_0, of 0.25 to 1.50 in steps of 0.25, from 1.5 to 4.0 in steps of 0.5, and values of 5 and 6. The behavior of these curves as a function of D/r_0 may be characterized as a constant value of unity for D/r_0 much smaller than unity, then a simple power-law decline to a second break where the value levels off to a constant value of $\exp\left[(\vartheta/\vartheta_0)^{5/3} \right]$.

In repeating this work for vertical propagation, daytime and night-time, for zenith angles from 0° to 70°, we have found no noticeable deviation from the results presented in Fig. 6, (i. e. , noticeable when the figures are overlayed). Accordingly we suggest that Fig. 6 can be considered to be a universal curve.

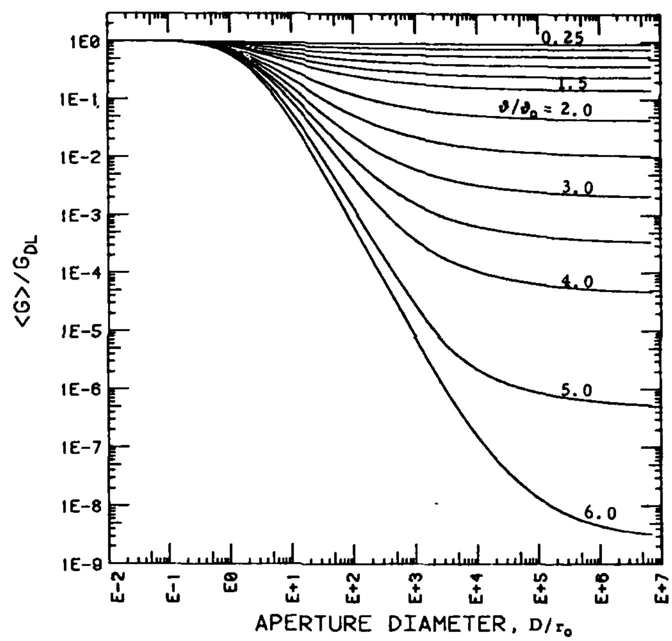

Figure 6. Normalized Laser Transmitter Antenna Gain.

Results are calculated for C_N^2 constant over the propagation path, but almost exactly match the results for day and for night vertical propagation at any zenith angle.

6. References for Chapter 1

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Appendix to Chapter 1

Computer Program Listing

The programs listed on the following pages was used to prepare the numerical data presented in this chapter.

```
C     PROGRAM UPLINK6
      REAL Q1(165),Y(165,5),Z(5)
      REAL*8 S(5),TERM1,TERM2,TERM3,TERM,SUM,E56,E53
      REAL*8 X,DX,Q,XX(100),X1(100),L0,L053
      E56 = 5./6.D0
      E53 = 5./3.D0
      E76 = 7./6.
      ALOG2 = ALOG(2.)
      N = 100
      DX = 1./N
      SUM1 = 0
      SUM2 = 0
      SUM3 = 0
      SUM4 = 0
      DO 50 I=1,N
      X = (I-.5)*DX
      XX(I) = X**E53
      X1(I) = (1.-X)**E53
      SUM1 = SUM1+DX*XX(I)
      SUM2 = SUM2+DX*X1(I)
      SUM3 = SUM3+DX*(1.-X)**(-1./3.)*X**2
      SUM4 = SUM4+DX*(1.-X)**2*X**(-1./3.)
50    CONTINUE
      THETA0 = (2.905*SUM1)**(-3./5.)
      R0 = (2.905/6.88*SUM2)**(-3./5.)
      L0 = R0/THETA0
      L053 = L0**E53
      ALPHA = (5./6.)*L0**(-1./3.)*SUM3/SUM1
      BETA = (5./6.)*L0**(1./3.)*SUM4/SUM2
      WRITE (6,60) THETA0,R0,L0,ALPHA,BETA
60    FORMAT (' THETA0 =',1PE12.4,3X,'R0 =',1PE12.4,3X,'L0 =',
     1  1PE12.4,3X,'ALPHA = ',1PE12.4,3X,'BETA =',1PE12.4///)
      DO 200 J=1,165
      Q = 2.**((J-85)/4.)
      Q1(J) = DLOG(Q)/ALOG2
      TERM3 = Q**E56
      DO 150 K=1,5
      C = (K-1)/4.
      SUM = 0
      DO 100 I=1,N
      X = (I-.5)*DX
      TERM1 = (1.-X)**2*Q+X**X/(Q*L0*L0)
      TERM2 = 2.*(1.-X)*X*C/L0
      TERM = X1(I)*TERM3+XX(I)/(TERM3*L053)
     1  -.5D0*((TERM1-TERM2)**E56+(TERM1+TERM2)**E56)
      SUM = SUM+TERM*DX
100   CONTINUE
      S(K) = 6.88/SUM2*SUM
      Y(J,K) = S(K)
      IF (Q.LT.1)  THEN
         Z(K) = 6.88*Q**E56*(1.-BETA*(1.-C*C/3.)*Q**(1./3.))
      ELSE
         Z(K) = Q**(-E56)*(1.-ALPHA*(1.-C*C/3.)*Q**(-1./3.))
         END IF
```

```
150    CONTINUE
       WRITE (6,160) Q1(J),(S(K),Z(K),K=1,5)
160    FORMAT (F7.2,1P,5(2X,2E10.2))
200    CONTINUE
       NPLOTS = 5
       DO 300 K=1,5
       CALL MPLTFL(100,165,Q1,Y(1,K),NPLOTS)
       NPLOTS = 0
300    CONTINUE
       END
```

```
C     PROGRAM UPLINK9
      REAL*4 S(165,5),Q(165),LG(55),K,LAMBDA
      REAL*4 SLOG(165,5),D(55),DG(55),DC(5)
      REAL*8 SUM
      LOGICAL IDONE
      PI = 3.14159265
      IFN = 103
      CALL OPENRD(IFN)
      DO 20 J=1,5
      READ(IFN) N,(Q(I),I=1,N),(S(I,J),I=1,N)
 20   CONTINUE
      CALL CLOSE(IFN,0)
      DO 30 J=1,5
      DO 30 I=1,164
 30   SLOG(I,J) = ALOG(S(I+1,J)/S(I,J))/ALOG(Q(I+1)/Q(I))
      DO 40 M=1,55
 40   D(M) = 10.**((M-13)/6.)
      DC(1) = ACOS(0.)-ACOS(.125)
      DO 50 J=1,4
 50   DC(J) = ACOS((J-1)/8.)-ACOS((J+1)/8.)
      DC(5) = ACOS(.875)-ACOS(1.)
      NPLOTS = 17
      NPTS = 200
      THETA = 0
      DO 200 J=1,17
      IF (THETA.LE.1.49)  THEN
         THETA = THETA+.25
      ELSE IF (THETA.LE.3.99)  THEN
         THETA = THETA+.5
      ELSE IF (THETA.LT.9.9)  THEN
         THETA = THETA+1.
      ELSE
         GO TO 200
         END IF
      DO 150 M=1,55
      DX0 = AMIN1(.001,.001/D(M))
      B = DX0**(-1./NPTS)
      SUM = (PI/2.)*(DX0/2.)
      X = 0
      IDONE = .FALSE.
      L0 = 165
      DO 100 I=1,NPTS
      X = X+DX0
      RHO = X*D(M)
      RT56 = (RHO*THETA)**(5./6.)
      SQX = SQRT(1-X*X)
      K = ATAN(SQX/X)-X*SQX
      Q1 = RHO/THETA
      DO 70 L=L0,1,-1
 70   IF (Q(L).LT.Q1)  GO TO 80
      L = 1
 80   CONTINUE
      IF (L.GT.164)  L=164
      L0 = L+1
```

```
      SUM1 = 0
      DO 90 J1=1,5
      S1 = S(L,J1)*(Q1/Q(L))**SLOG(L,J1)
      S1 = S1*RT56
      IF (S1.LT.150)  SUM1 = SUM1+EXP(-S1)*DC(J1)
90    CONTINUE
      DX1 = DX0*B
      IF ((X+DX1).GT.1)  THEN
         DX1 = 1.-X
         IDONE = .TRUE.
         END IF
      SUM = SUM+(4*K*SUM1)*(DX0+DX1)/2.*X
      IF (IDONE)  GO TO 110
      DX0 = DX1
100   CONTINUE
      WRITE (0,105)  J,M,X+DX1
105   FORMAT (' INTEGRAL NOT DONE FOR J,M,X =',2I7,F10.3)
110   CONTINUE
      LG(M) = 2./PI*SUM/(PI/4.)
150   CONTINUE
      CALL MPLTFL(110,55,D,LG,NPLOTS)
      NPLOTS = 0
200   CONTINUE
      END
      END
      END
      END
```

```
C       PROGRAM UPLINK10
        REAL Q1(165),Y(165,5),DX(300),CN2(300),A(6),CN(6)
        REAL*8 S(5),TERM1,TERM2,TERM3,TERM,SUM,E56,E53
        REAL*8 X,Q,XX(300),X1(300),X0(300),L0,L0R53,L0R
        REAL Z(165,5)
        LOGICAL*1 IDONE
        E56 = 5./6D0
        E53 = 5./3D0
        E76 = 7./6
4       WRITE (0,5)
5       FORMAT (' DAY OR NIGHT?')
        READ (0,6) ANS
6       FORMAT (A1)
        IF (ANS.EQ.'D')  THEN
           IFN = 90
           JFN = 104
        ELSE IF (ANS.EQ.'N')  THEN
           IFN = 91
           JFN = 105
        ELSE
           GO TO 4
           END IF
        CALL OPENRD (IFN)
        READ (IFN) N,(A(I),I=1,N),(CN(I),I=1,N)
        CALL CLOSE(IFN,0)
        N = 200
        R = 20.E5
        DX0 = 1.E-7
        DX00 = DX0
        B = (.01/DX0)**(1./N)
        J1 = 1
        J2 = 4
        SUM1 = 0
        SUM2 = 0
        SUM3 = 0
        SUM4 = 0
        X = -DX0/2.
        DO 50 I=1,N
        X = X+DX0
        X0(I) = X
        XX(I) = X**E53
        X1(I) = (1.-X)**E53
        H = X*R+A(1)
        DO 20 J=J1,J2
20      IF (A(J).LT.H.AND.A(J+1).GE.H)  GO TO 25
        GO TO 60
        GO TO 30
25      VAL = LOG(CN(J))+LOG(H/A(J))/LOG(A(J+1)/A(J))*LOG(CN(J+1)/CN(J))
        CN2(I) = EXP(VAL)
30      J1 = J
        DX1 = DX0*B
        DX(I) = (DX0+DX1)/2.
        CN2DX = CN2(I)*DX(I)
        SUM1 = SUM1+CN2DX*XX(I)
```

```
      SUM2 = SUM2+CN2DX*X1(I)
      SUM3 = SUM3+CN2DX*(1.-X)**(-1./3.)*X**2
      SUM4 = SUM4+CN2DX*(1.-X)**2*X**(-1./3.)
      DX0 = DX1
50    CONTINUE
60    CONTINUE
      THETA0 = (2.905*SUM1*R**(8./3.))**(-3./5.)
      R0 = (2.905/6.88*SUM2*R)**(-3./5.)
      L0 = R0/THETA0
      L0R = L0/R
      L0RS3 = L0R**E53
      ALPHA = (5./6.)*(L0/R)**(-1./3.)*SUM3/SUM1
      BETA = (5./6.)*(L0/R)**(1./3.)*SUM4/SUM2
      WRITE (6,61)  THETA0,R0,L0,ALPHA,BETA
61    FORMAT (' THETA0 =',1PE12.4,3X,'R0 =',1PE12.4,3X,'L0 =',
     1   1PE12.4,3X,'ALPHA = ',1PE12.4,3X,'BETA =',1PE12.4///)
      IMAX = I-1
      WRITE (0,70) IMAX
70    FORMAT (I5,' STEPS ARE USED IN EACH INTEGRAL.')
C     DO 65 I=1,IMAX
C     WRITE (6,66) X0(I),DX(I),X0(I)*R+A(1),CN2(I)
65    CONTINUE
66    FORMAT (1P4E20.5)
      DO 200 J=1,165
      Q = 2.**((J-85)/4.)
      Q1(J) = Q
      TERM3 = Q**E56
      DO 150 K=1,5
      C = (K-1)/4.
      SUM = 0
      DO 100 I=1,IMAX
      X = X0(I)
      TERM1 = (1.-X)**2*Q+X*X/(Q*L0R*L0R)
      TERM2 = 2.*(1.-X)*X*C/L0R
      TERM = X1(I)*(TERM3+XX(I)/(TERM3*L0RS3)
     1   -.5D0*((TERM1-TERM2)**E56+(TERM1+TERM2)**E56)
      SUM = SUM+TERM*DX(I)*CN2(I)
100   CONTINUE
      S(K) = 6.88/SUM2*SUM
      Y(J,K) = S(K)
      IF (Q.LT..2)  THEN
         Z(J,K) = 6.88*Q**E56*(1.-BETA*(1.-C*C/3.)*Q**(1./3.))
      ELSE
         Z(J,K) = Q**(-E56)*(1.-ALPHA*(1.-C*C/3.)*Q**(-1./3.))
         END IF
150   CONTINUE
      WRITE (6,160) Q,(S(K),Z(J,K),K=1,5)
160   FORMAT (1PE10.2,5(2X,2E10.2))
200   CONTINUE
      NPLOTS = 5
      DO 300 K=1,5
      CALL MPLTFL(JFN,165,Q1,Y(1,K),NPLOTS)
      NPLOTS = 0
300   CONTINUE
      NPLOTS = 2
```

```
      DO 400 K=1,5,4
      IF (K.EQ.1)  THEN
         N1 = 75
         N2 = 90
      ELSE
         N1 = 75
         N2 = 90
         END IF
      I1 = 165-N2+1
      CALL MPLTFL(JFN+2,165,Q1,Y(1,K),NPLOTS)
      CALL MPLTFL(JFN+4,N1,Q1,Z(1,K),3*NPLOTS)
      CALL MPLTFL(JFN+4,N2,Q1(I1),Z(I1,K),0)
      NPLOTS = 0
400   CONTINUE
      END
      END
```

```
C     PROGRAM UPLINK10
      REAL Q1(165),Y(165,5),DX(300),CN2(300),A(6),CN(6)
      REAL*8 S(5),TERM1,TERM2,TERM3,TERM,SUM,E56,E53
      REAL*8 X,Q,XX(300),X1(300),X0(300),L0,L0R53,L0R
      REAL Z(165,5)
      LOGICAL*1 IDONE
      E56 = 5./6D0
      E53 = 5./3D0
      E76 = 7./6
4     WRITE (0,5)
5     FORMAT (' DAY OR NIGHT?')
      READ (0,6) ANS
6     FORMAT (A1)
      IF (ANS.EQ.'D')  THEN
         IFN = 90
         JFN = 104
      ELSE IF (ANS.EQ.'N')  THEN
         IFN = 91
         JFN = 105
      ELSE
        GO TO 4
        END IF
      CALL OPENRD (IFN)
      READ (IFN) N,(A(I),I=1,N),(CN(I),I=1,N)
      CALL CLOSE(IFN,0)
      N = 200
      R = 20.E5
      DX0 = 1.E-7
      DX00 = DX0
      B = (.01/DX0)**(1./N)
      J1 = 1
      J2 = 4
      SUM1 = 0
      SUM2 = 0
      SUM3 = 0
      SUM4 = 0
      X = -DX0/2.
      DO 50 I=1,N
      X = X+DX0
      X0(I) = X
      XX(I) = X**E53
      X1(I) = (1.-X)**E53
      H = X*R+A(1)
      DO 20 J=J1,J2
20    IF (A(J).LT.H.AND.A(J+1).GE.H)  GO TO 25
      GO TO 60
      GO TO 30
25    VAL = LOG(CN(J))+LOG(H/A(J))/LOG(A(J+1)/A(J))*LOG(CN(J+1)/CN(J))
      CN2(I) = EXP(VAL)
30    J1 = J
      DX1 = DX0*B
      DX(I) = (DX0+DX1)/2.
      CN2DX = CN2(I)*DX(I)
      SUM1 = SUM1+CN2DX*XX(I)
```

```
      SUM2 = SUM2+CN2DX*X1(I)
      SUM3 = SUM3+CN2DX*(1.-X)**(-1./3.)*X**2
      SUM4 = SUM4+CN2DX*(1.-X)**2*X**(-1./3.)
      DX0 = DX1
 50   CONTINUE
 60   CONTINUE
      THETA0 = (2.905*SUM1*R**(8./3.))**(-3./5.)
      R0 = (2.905/6.88*SUM2*R)**(-3./5.)
      L0 = R0/THETA0
      L0R = L0/R
      L0R53 = L0R**E53
      ALPHA = (5./6.)*(L0/R)**(-1./3.)*SUM3/SUM1
      BETA = (5./6.)*(L0/R)**(1./3.)*SUM4/SUM2
      WRITE (6,61)  THETA0,R0,L0,ALPHA,BETA
 61   FORMAT (' THETA0 =',1PE12.4,3X,'R0 =',1PE12.4,3X,'L0 =',
     1   1PE12.4,3X,'ALPHA = ',1PE12.4,3X,'BETA =',1PE12.4///)
      IMAX = I-1
      WRITE (0,70) IMAX
 70   FORMAT (I5,' STEPS ARE USED IN EACH INTEGRAL.')
C     DO 65 I=1,IMAX
C     WRITE (6,66) X0(I),DX(I),X0(I)*R+A(1),CN2(I)
 65   CONTINUE
 66   FORMAT (1P4E20.5)
      DO 200 J=1,165
      Q = 2.**((J-85)/4.)
      Q1(J) = Q
      TERM3 = Q**E56
      DO 150 K=1,5
      C = (K-1)/4.
      SUM = 0
      DO 100 I=1,IMAX
      X = X0(I)
      TERM1 = (1.-X)**2*Q+X*X/(Q*L0R*L0R)
      TERM2 = 2.*(1.-X)*X*C/L0R
      TERM = X1(I)*TERM3+XX(I)/(TERM3*L0R53)
     1   -.5D0*((TERM1-TERM2)**E56+(TERM1+TERM2)**E56)
      SUM = SUM+TERM*DX(I)*CN2(I)
100   CONTINUE
      S(K) = 6.88/SUM2*SUM
      Y(J,K) = S(K)
      IF (Q.LT..2)  THEN
         Z(J,K) = 6.88*Q**E56*(1.-BETA*(1.-C*C/3.)*Q**(1./3.))
      ELSE
         Z(J,K) = Q**(-E56)*(1.-ALPHA*(1.-C*C/3.)*Q**(-1./3.))
         END IF
150   CONTINUE
      WRITE (6,160) Q,(S(K),Z(J,K),K=1,5)
160   FORMAT (1PE10.2,5(2X,2E10.2))
200   CONTINUE
      NPLOTS = 5
      DO 300 K=1,5
      CALL MPLTFL(JFN,165,Q1,Y(1,K),NPLOTS)
      NPLOTS = 0
300   CONTINUE
      NPLOTS = 2
```

```
      DO 400 K=1,5,4
      IF (K.EQ.1)  THEN
         N1 = 75
         N2 = 90
      ELSE
         N1 = 75
         N2 = 90
         END IF
      I1 = 165-N2+1
      CALL MPLTFL(JFN+2,165,Q1,Y(1,K),NPLOTS)
      CALL MPLTFL(JFN+4,N1,Q1,Z(1,K),3*NPLOTS)
      CALL MPLTFL(JFN+4,N2,Q1(I1),Z(I1,K),0)
      NPLOTS = 0
400   CONTINUE
      END
      END
```

```
C     PROGRAM UPLINK11
      REAL*4 S(165,5),Q(165),LG(55),K,LAMBDA
      REAL*4 SLOG(165,5),D(55),DG(55),DC(5)
      REAL*8 SUM
      LOGICAL IDONE
      PI = 3.14159265
4     WRITE (0,5)
5     FORMAT (' DAY OR NIGHT?')
      READ (0,6) ANS
6     FORMAT (A1)
      IF (ANS.EQ.'D')  THEN
         IFN = 104
         JFN = 111
      ELSE IF (ANS.EQ.'N')  THEN
         IFN = 105
         JFN = 112
      ELSE
         GO TO 4
         END IF
      CALL OPENRD(IFN)
      DO 20 J=1,5
      READ(IFN) N,(Q(I),I=1,N),(S(I,J),I=1,N)
20    CONTINUE
      CALL CLOSE(IFN,0)
      DO 30 J=1,5
      DO 30 I=1,164
30    SLOG(I,J) = ALOG(S(I+1,J)/S(I,J))/ALOG(Q(I+1)/Q(I))
      DO 40 M=1,55
40    D(M) = 10.**((M-13)/6.)
      DC(1) = ACOS(0.)-ACOS(.125)
      DO 50 J=1,4
50    DC(J) = ACOS((J-1)/8.)-ACOS((J+1)/8.)
      DC(5) = ACOS(.875)-ACOS(1.)
      NPLOTS = 17
      NPTS = 200
      THETA = 0
      DO 200 J=1,17
      IF (THETA.LE.1.49)  THEN
         THETA = THETA+.25
      ELSE IF (THETA.LE.3.99)  THEN
         THETA = THETA+.5
      ELSE IF (THETA.LT.9.9)  THEN
         THETA = THETA+1.
      ELSE
         GO TO 200
         END IF
      DO 150 M=1,55
      DX0 = AMIN1(.001,.001/D(M))
      B = DX0**(-1./NPTS)
      SUM = 0
      X = -DX0/2.
      IDONE = .FALSE.
      L0 = 165
      DO 100 I=1,NPTS
```

```
      X = X+DX0
      RHO = X*D(M)
      RT56 = (RHO*THETA)**(5./6.)
      SQX = SQRT(1-X*X)
      K = ATAN(SQX/X)-X*SQX
      Q1 = RHO/THETA
      DO 70 L=L0,1,-1
70    IF (Q(L).LT.Q1)  GO TO 80
      L = 1
80    CONTINUE
      IF (L.GT.164)  L=164
      L0 = L+1
      SUM1 = 0
      DO 90 J1=1,5
      S1 = S(L,J1)*(Q1/Q(L))**SLOG(L,J1)
      S1 = S1*RT56
      IF (S1.LT.150)  SUM1 = SUM1+EXP(-S1)*DC(J1)
90    CONTINUE
      DX1 = DX0*B
      IF ((X+DX1).GT.1)  THEN
         DX1 = 1.-X
         IDONE = .TRUE.
         END IF
      SUM = SUM+(4*K*SUM1)*(DX0+DX1)/2.*X
      IF (IDONE)  GO TO 110
      DX0 = DX1
100   CONTINUE
      WRITE (0,105)  J,M,X+DX1
105   FORMAT (' INTEGRAL NOT DONE FOR J,M,X =',2I7,F10.3)
110   CONTINUE
      LG(M) = 2./PI*SUM/(PI/4.)
150   CONTINUE
      CALL MPLTFL(JFN,55,D,LG,NPLOTS)
      NPLOTS = 0
200   CONTINUE
      END
      END
      END
      END
```

```
C     PROGRAM UPLINK12
      REAL*4 S(165,5),Q(165),LG(31),K,LAMBDA
      REAL*4 SLOG(165,5),D(31),DG(31),DC(5),ANS(2)
      REAL*8 SUM
      LOGICAL IDONE
      PI = 3.14159265
4     WRITE (0,5)
5     FORMAT (' DAY OR NIGHT?')
      READ (0,6) IANS
6     FORMAT (A1)
      IF (IANS.EQ.'D')  THEN
         IFN = 104
         JFN = 113
         TH0 = 4.1875E3
         R0 = 1.6663E7
      ELSE IF (IANS.EQ.'N')  THEN
         IFN = 105
         JFN = 114
         TH0 = 4.6037E3
         R0 = 3.3694E7
      ELSE
         GO TO 4
         END IF
      CALL OPENRD(IFN)
      DO 20 J=1,5
      READ(IFN) N,(Q(I),I=1,N),(S(I,J),I=1,N)
20    CONTINUE
      CALL CLOSE(IFN,0)
      DO 30 J=1,5
      DO 30 I=1,164
30    SLOG(I,J) = ALOG(S(I+1,J)/S(I,J))/ALOG(Q(I+1)/Q(I))
      WRITE (0,32)
32    FORMAT (' ENTER WAVELENGTH & ZENITH ANGLE')
      CALL INPUT(ANS,2)
      LAMBDA = ANS(1)
      ZA = ANS(2)*PI/180.
      COSZA = COS(ZA)
      SECZA = 1./COSZA
      R = 20.E5*SECZA
      WAVNUM = 2*PI/LAMBDA
      R0 = R0*(WAVNUM**2*SECZA)**(-3./5.)
      TH0 = TH0*(WAVNUM**2)**(-3./5.)*SECZA**(-8./5.)
      WRITE (0,35) LAMBDA,ZA,R0,TH0
35    FORMAT (' LAMBDA =',1PE10.2,5X,'ZA =',0PF6.2,5X,'R0 =',F10.4,
     1    5X,'THETA0 =',1PE10.2)
      DO 40 M=1,31
40    D(M) = 10.**((M-11)/10.)
      DC(1) = ACOS(0.)-ACOS(.125)
      DO 50 J=1,4
50    DC(J) = ACOS((J-1)/8.)-ACOS((J+1)/8.)
      DC(5) = ACOS(.875)-ACOS(1.)
      NPLOTS = 17
      NPTS = 200
      TH = 0
```

```
      DO 200 J=1,17
      IF (TH.LE.57E-6)  THEN
         TH = TH+5E-6
      ELSE
         GO TO 200
         END IF
      THETA = TH/THO
      DO 150 M=1,31
      DRO = D(M)/RO
      DXO = AMIN1(.001,.001/DRO)
      B = DXO**(-1./NPTS)
      SUM = 0
      X = -DXO/2.
      IDONE = .FALSE.
      LO = 165
      DO 100 I=1,NPTS
      X = X+DXO
      RHO = X*DRO
      RT56 = (RHO*THETA)**(5./6.)
      SQX = SQRT(1-X*X)
      K = ATAN(SQX/X)-X*SQX
      Q1 = RHO/THETA
      DO 70 L=LO,1,-1
70    IF (Q(L).LT.Q1)  GO TO 80
      L = 1
80    CONTINUE
      IF (L.GT.164)  L=164
      LO = L+1
      SUM1 = 0
      DO 90 J1=1,5
      S1 = S(L,J1)*(Q1/Q(L))**SLOG(L,J1)
      S1 = S1*RT56
      IF (S1.LT.150)  SUM1 = SUM1+EXP(-S1)*DC(J1)
90    CONTINUE
      DX1 = DXO*B
      IF ((X+DX1).GT.1)  THEN
         DX1 = 1.-X
         IDONE = .TRUE.
         END IF
      SUM = SUM+(4*K*SUM1)*(DXO+DX1)/2.*X
      IF (IDONE)  GO TO 110
      DXO = DX1
100   CONTINUE
      WRITE (0,105)  J,M,X+DX1
105   FORMAT (' INTEGRAL NOT DONE FOR J,M,X =',2I7,F10.3)
110   CONTINUE
      LG(M) = 2./PI*SUM/(PI/4.)
150   CONTINUE
      CALL MPLTFL(JFN,31,D,LG,NPLOTS)
200   CONTINUE
      END
      ENDFILE JFN
      END
```

Chapter 2

Pulse Laser Backscatter

The analysis begins with the equation that describes the reduction in the energy associated with the laser pulse as it propagates through the scattering medium

$$E(z_2) = E(z_1) \exp \left[-\int_{z_1}^{z_2} dz \, \beta(z) \right] \quad . \tag{1}$$

Here E(z) is the energy in the pulse at altitude, z, and β(z) is the total scattering coefficient, which is assumed to be a function of altitude. The total energy scattered out of the laser beam, E_s, is then given by

$$E_s = E(z_1) \left\{ 1 - \exp\left[-\int_{z_1}^{z_2} dz \, \beta(z) \right] \right\} \quad . \tag{2}$$

We are interested in the case when $E(z_1) = 1$ joule. Therefore, only the quantity, β(z), need be evaluated to determine the total energy scattered out of the beam.

We assume that the only scattering mechanism is Rayleigh scattering by air molecules. Thus, an evaluation of Rayleigh scattering will lead to a determination of β(z) . When it is assumed that the scattering is incoherent, the angular scattering coefficient, β(ϕ), is given by[1]

$$\beta(\phi) = \frac{\pi^2 (n^2 - 1)^2}{N \lambda^4} \sin^2 \phi \quad , \tag{3}$$

where ϕ is the scattering angle and equals 90° in the forward direction, and 270° is the backward direction. In addition, n is the refractive index, N is the number of molecules per unit volume, and λ is the wavelength. The total scattering coefficient, β, is obtained by integrating β(ϕ) over the solid angle

$$\beta = \int d\Omega \, \beta(\phi) \quad . \tag{4}$$

We recognize that $\int d\Omega = 2\pi \int_0^\pi d\phi \, \sin\phi$ and Eq. (4) becomes

$$\beta = \frac{2\pi^3 (n^2 - 1)^2}{N \lambda^4} \int_0^\pi d\phi \, \sin^3\phi \quad . \tag{5}$$

The integral is a standard form and equals 4/3, therefore, the total scattering coefficient is given by the expression

$$\beta = \frac{8\pi^3 (n^2 - 1)^2}{3N \lambda^4} \quad . \tag{6}$$

We will be interested in the scattering mechanism over a wide range of altitudes and for this reason it is necessary to determine how β varies with increasing altitude. The quantity $n^2 - 1$ is proportional to N, the number of molecules per unit volume. Therefore, β is proportional to N. From the perfect gas law N is proportional to pressure and inversely proportional to temperature. Therefore, if β is known at one altitude, its value at other altitudes is given by

$$\beta(z) = \frac{8\pi^3 (n^2 - 1)^2}{3N \lambda^4} \, \frac{P(z)}{P_o} \, \frac{T_o}{T(z)} \quad , \tag{7}$$

where P_o and T_o are the original pressure and temperature at which the total scattering coefficient is known, and P(z) and T(z) are the pressure and temperature at the desired altitude. The total energy scattered out of the beam is then

$$E_s = E(z_1) \left\{ 1 - \exp\left[- \frac{8\pi^3 (n^2 - 1)^2}{3N \lambda^4} \, \frac{T_o}{P_o} \int_{z_1}^{z_2} dz \, \frac{P(z)}{T(z)} \right] \right\} \quad . \tag{8}$$

When it is assumed that this energy is uniformly distributed over a sphere centered at an altitude of $(z_1 + z_2)/2$ the energy incident on a receiver with area of 1 m^2 is obtained by multiplying E_s by the solid angle subtended by the receiver. The energy received, E_R, is then

$$E_R = E_s / \left(\frac{z_1 + z_2}{2}\right)^2 \quad . \tag{9}$$

We are mainly concerned with wavelengths in the visible regions of the spectrum where photon counting detectors such as photomultipliers are available. It is, therefore, more appropriate to determine the number of photons, N_p, collected by the receiver aperture. The energy per photon is equal to hc/λ where h is Planck's constant and c is the speed of light. Thus,

$$N_p = \frac{E_s \lambda}{hc \left(\frac{z_1 + z_2}{2}\right)^2} \quad . \tag{10}$$

Eq. (10) along with Eq. (8) are the primary results of the analysis, the total energy scattered out of the laser beam and the total number of photons collected by the receiver aperture can both be determined. The results for various wavelengths is the visible region of the spectrum, assuming a one joule incident pulse, are summarized in Fig.'s 1 and 2. To obtain these results the following values of the parameters were used

$$n = 1.000293 \quad , \tag{11a}$$

$$N = 2.547 \times 10^{25} \text{ m}^{-3} \quad , \tag{11b}$$

$$P_o = 1013.25 \text{ mb} \quad , \tag{11c}$$

$$T_o = 288.15^\circ \text{K} \quad , \tag{11d}$$

λ (μ)	z_1 = 10 km, z_2 = 20 km		z_1 = 15 km, z_2 = 25 km		z_1 = 20 km, z_2 = 30 km	
	Fractional Power Loss into 4π steradians	Number Photons into 1m^2 aperture/ joule of Incident Beam	Fractional Power Loss into 4π steradians	Number Photons into 1m^2 aperture/ joule of Incident Beam	Fractional Power Loss into 4π steradians	Number Photons into 1m^2 aperture/ joule of Incident Beam
1	1.9642×10^{-3}	4.3890×10^{7}	8.9402×10^{-4}	1.1237×10^{7}	4.0440×10^{-4}	3.2531×10^{6}
.7	8.1728×10^{-3}	1.2783×10^{8}	3.7219×10^{-3}	3.2747×10^{7}	1.6840×10^{-3}	9.4826×10^{6}
.6	1.5256×10^{-2}	2.0454×10^{8}	6.9190×10^{-3}	5.2179×10^{7}	3.1198×10^{-3}	1.5058×10^{7}
.5	3.1895×10^{-2}	3.5635×10^{8}	1.4298×10^{-2}	8.9857×10^{7}	6.4901×10^{-3}	2.6104×10^{7}
.4	7.9667×10^{-2}	7.1206×10^{8}	3.5523×10^{-2}	1.7860×10^{8}	1.5919×10^{-2}	5.1223×10^{7}
.3	3.7412×10^{-1}	1.8376×10^{9}	1.1032×10^{-1}	4.1599×10^{8}	5.1183×10^{-2}	1.2352×10^{8}

Figure 1. Atmospheric Scattering Results for Three Different Altitude Ranges.

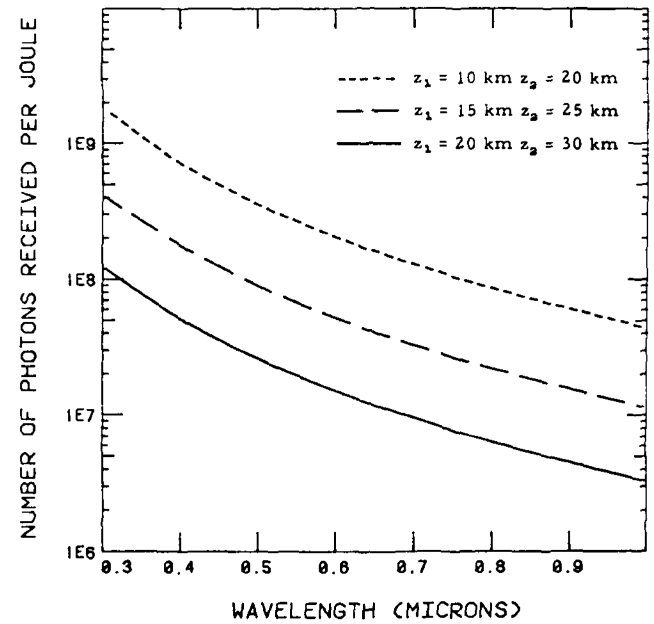

Figure 2. The Number of Photons Incident Upon a 1m^2 Receiver Area for a One Joule Incident Pulse.

These results ignore atmospheric attenuation between the receiver and altitude z_1. Vertical propagation is assumed.

where the values for the first two quantities are appropriate for air at sea level temperature and pressure conditions, and were obtained from Ref. 1. In addition to P_0 and T_0, the values of $P(z)$ and $T(z)$ used to evaluate the integral in Eq. (8) were obtained from Ref. 2 .

Fig. 1 summarizes the results for the actual wavelengths that were used. Fig. 2 is a plot of Eq. (10) for each of the three altitude ranges considered, indicating that for a wavelength of 1 μ over three million photons are available if the atmospheric scattering is observed from 20 to 30 kilometers in altitude. This is a significant number and can easily be detected with existing devices.

References for Chapter 2

1. Earl J. McCartney, Optics of the Atmosphere, John Wiley and Sons (New York 1976), Chapter 4.

2. U.S. Standard Atmosphere Supplements, 1966, Prepared under the sponsorship of Environmental Science Services Administration, National Aeronautics and Space Administration, and the United States Air Force. Table 5.1, Mid-Latitudes, Spring/Fall, p 118.

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